Relation between pentagonal and GS-quasigroups

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Definition

A **quasigroup** $(Q, \cdot)$ *is a grupoid in which for every* $a, b \in Q$ *there exist unique* $x, y \in Q$ *such that* $a \cdot x = b$ *and* $y \cdot a = b$*.*

To make some expressions shorter and more readable we use abbreviations. For example, instead of writing $a \cdot ((b \cdot c) \cdot d)$ we write $a(bc \cdot d)$.

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Definition

An **IM-quasigroup** *is a quasigroup* $(Q, \cdot)$ *in which following properties hold:*

- $a \cdot a = a \quad \forall a \in Q$ **idempotency**
- $ab \cdot cd = ac \cdot bd \quad \forall a, b, c, d \in Q$ **mediality**

Along with idempotency and mediality, in IM-quasigroups next three properties are valid:

- $ab \cdot a = a \cdot ba \quad \forall\, a, b \in Q$ **elasticity**
- $ab \cdot c = ac \cdot bc \quad \forall\, a, b, c \in Q$ **right distributivity**
- $a \cdot bc = ab \cdot ac \quad \forall\, a, b, c \in Q$ **left distributivity**

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Example

$C(q) = (\mathbb{C}, *)$, *where* $*$ *is defined with*

$$a * b = (1 - q)a + qb,$$

and $q \in \mathbb{C}$, $q \neq 0, 1$.

Definition

A **GS-quasigroup** *is a quasigroup* $(Q, \cdot)$ *in which following properties hold:*

- $a \cdot a = a \quad \forall\, a \in Q$ **idempotency**
- $a(ab \cdot c) \cdot c = b \quad \forall\, a, b, c \in Q$

- every GS-quasigroup is an IM-quasigroup

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$C(q) = (\mathbb{C}, *)$, *where* $*$ *is defined with*

$$a * b = (1 - q)a + qb,$$

and q *is a solution of the equation* $q^2 - q - 1 = 0$.

Solutions of the equation $q^2 - q - 1 = 0$ are

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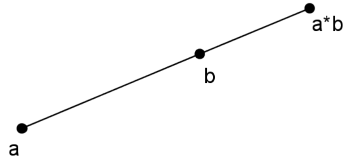
If we regard the complex numbers as the points of the Euclidean plane and if we rewrite $a * b = (1-q)a + qb$ as

$$\frac{a*b-a}{b-a} = q,$$

we notice that the point $a * b$ divides the pair a, b in the ratio q, i.e. golden-section ratio.



Definition

A **pentagonal quasigroup** *is an IM-quasigroup* $(Q, \cdot)$ *in which following property holds*

- $(ab \cdot a)b \cdot a = b \quad \forall\, a, b \in Q$ **pentagonality**

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All calculations in pentagonal quasigroups are done using properties of idempotency, mediality, elasticity, left and right distributivity and following properties (which all arise from pentagonality):

- $(ab \cdot a)b \cdot a = b \quad \forall\, a, b \in Q$ (1)
- $(ab \cdot a)c \cdot a = bc \cdot b \quad \forall\, a, b, c \in Q$ (2)
- $(ab \cdot a)a \cdot a = ba \cdot b \quad \forall\, a, b \in Q$ (3)
- $ab \cdot (ba \cdot a)a = b \quad \forall\, a, b \in Q$ (4)
- $a(b \cdot (ba \cdot a)a) \cdot a = b \quad \forall\, a, b \in Q$ (5)

Theorem

In an IM-quasigroup $(Q, \cdot)$ identities (1), (2), (3) and (4) are all mutually equivalent and they imply identity (5).

Example

$C(q) = (\mathbb{C}, *)$, *where* $*$ *is defined with*

$$a * b = (1 - q)a + qb,$$

and q *is a solution of the equation* $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$.

This equation arises from the property of pentagonality.

Solutions of the equation $q^4 - 3q^3 + 4q^2 - 2q + 1 = 0$ are:

$$q_{1,2} = \frac{1}{4}(3 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}) \approx 1.31 \pm 0.95i$$

$$q_{3,4} = \frac{1}{4}(3 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}) \approx 0.19 \pm 0.59i$$

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If we regard the complex numbers as the points of the Euclidean plane and if we rewrite $a * b = (1 - q)a + qb$ as

$$\frac{a * b - a}{b - a} = \frac{q - 0}{1 - 0},$$

we notice that points a, b and $a * b$ are the vertices of a triangle directly similar to the triangle with the vertices 0, 1 and q.

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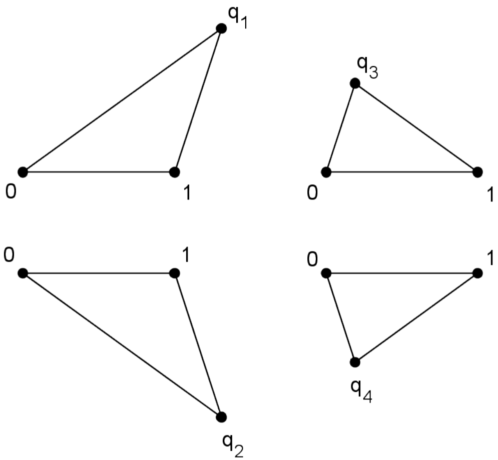
we notice that points a, b and $a * b$ are the vertices of a triangle directly similar to the triangle with the vertices 0, 1 and q.
We get a **characteristic triangle** for each of q_i, $i = 1, 2, 3, 4$.


q1
0
1
q3
0
1
0
1
q2
0
1
q4

A more general example of GS / pentagonal quasigroups is $(Q, *)$,

$$a * b = a + \varphi(b - a),$$

where $(Q, +)$ is an abelian group and φ is its automorphism which satisfies $\varphi^2 - \varphi - \mathbb{1} = 0$ / $\varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + \mathbb{1} = 0$.

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It can be shown that these are in fact the most general examples of GS / pentagonal quasigroups. We get Toyoda-like representation theorems for them.

Theorem

GS-quasigroup on the set Q exists if and only if exists an abelian group on the set Q with an automorphism φ which satisfies

$$\varphi^2 - \varphi - \mathbb{1} = 0.$$

Theorem

Pentagonal quasigroup on the set Q exists if and only if exists an abelian group on the set Q with an automorphism φ which satisfies

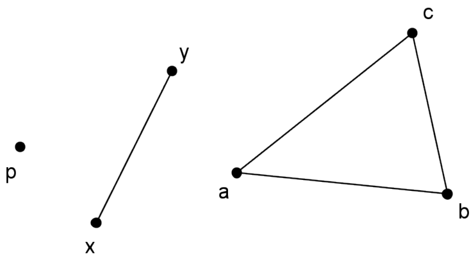
$$\varphi^4 - 3\varphi^3 + 4\varphi^2 - 2\varphi + \mathbb{1} = 0.$$

Contents

Let's first introduce some basic geometric concepts.

Definition

A **point** *in the quasigroup* $(Q, \cdot)$ *is an element of the set* Q.
A **segment** *in the quasigroup* $(Q, \cdot)$ *is a pair of points* $\{a, b\}$.
A n-**gon** *in the quasigroup* $(Q, \cdot)$ *is an ordered* n*-tuple of points* $(a_1, a_2, \ldots, a_n)$ *up to a cyclic permutation.*



Geometry of pentagonal quasigroups

- parallelogram, midpoint of the segment, center of the parallelogram

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- midpoint doesn't have to be unique: quasigroup Q_{16} with 16 elements

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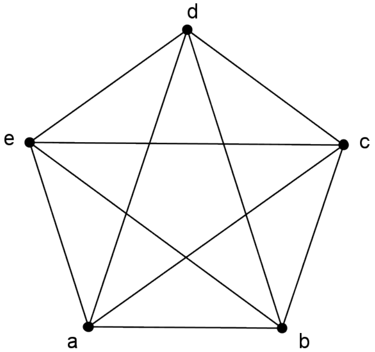
Definition

Let a, b, c, d and e be points of a pentagonal quasigroup $(Q,\cdot)$. Pentagon (a,b,c,d,e) is called **regular pentagon** *if $ab=c$, $bc=d$ and $cd=e$. This is denoted by $RP(a,b,c,d,e)$.*

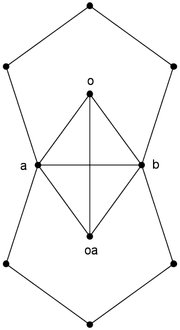


Theorem

A regular pentagon (a, b, c, d, e) *is uniquely determined by the ordered pair of points* (a, b).

Definition

Let a, b, c, d and e be points in a pentagonal quasigroup $(Q, \cdot)$ *such that* $RP(a, b, c, d, e)$*. The* **center of the regular pentagon** (a, b, c, d, e) *is the point o such that* $o = oa \cdot b$.



If we rewrite $o = oa \cdot b$ using theorem of characterization, we get

$$(2 \cdot \mathbb{1} - \varphi)(o) = (\mathbb{1} - \varphi)(a) + b.$$

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Example

$(Q_5, \cdot)$, $RP(0, 1, 2, 3, 4)$

$\cdot$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

$00 \cdot 1 = 2$, $10 \cdot 1 = 3$, $20 \cdot 1 = 4$, $30 \cdot 1 = 0$, $40 \cdot 1 = 1$

There is no o such that $o = oa \cdot b$.

Quasigroup $(Q_5, \cdot)$ is generated by the automorphism $\varphi(x) = 2x$.

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- DGS-trapezoids, GS-deltoids, affine regular dodecachedron, affine regular icosahedron...

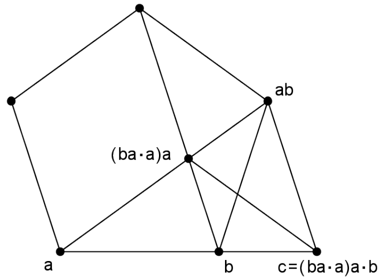
Contents

Theorem

Let $(Q, \cdot)$ be a pentagonal quasigroup and let $*\colon Q \times Q \to Q$ be a binary operation defined with

$$a * b = (ba \cdot a)a \cdot b.$$

Then $(Q, *)$ is GS-quasigroup.



Previous theorem tells that pentagonal quasigroup "inherits" entire geometry of GS-quasigroups.

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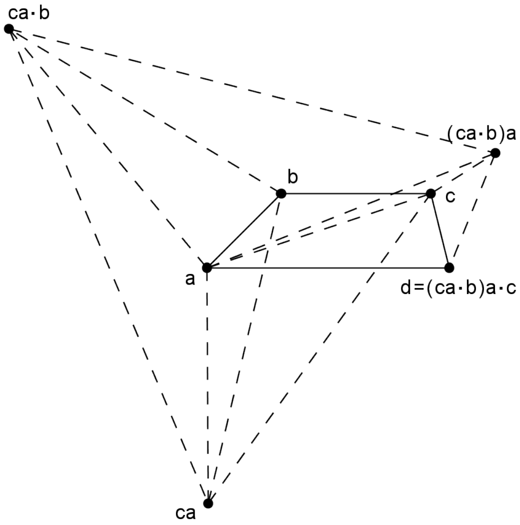
GS-trapezoid (a, b, c, d) is defined in GS-quasigroup and it is completely determined with its three vertices a, b and c. Previous theorem enables definition of GS-trapezoid in any pentagonal quasigroup.

Definition

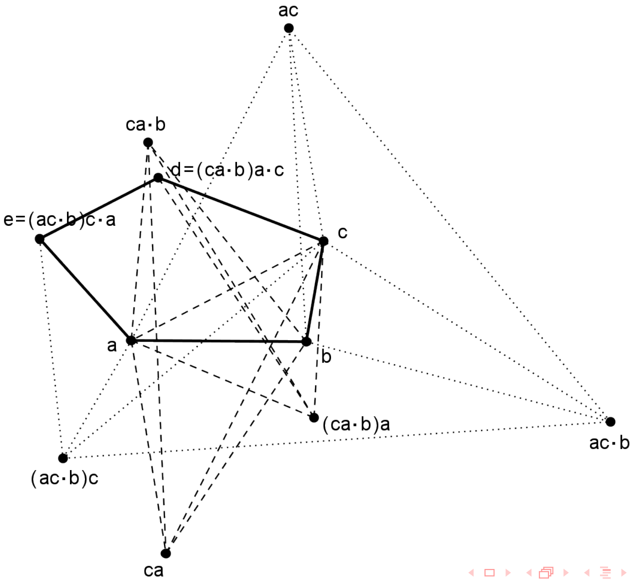
Let $(Q, \cdot)$ be a pentagonal quasigroup and a, b, c, $d \in Q$. We say that quadrangle (a, b, c, d) is **GS-trapezoid**, *denoted by $GST(a, b, c, d)$, if $d = (ca \cdot b)a \cdot c$.*


ca·b
(ca·b)a
b
c
a
d=(ca·b)a·c
ca

Concept of **affine regular pentagon** (a, b, c, d, e) is defined in GS-quasigroup if (a, b, c, d) and (b, c, d, e) are GS-trapezoids. It is completely determined with its three vertices a, b and c. Previous theorem enables definition of affine regular pentagon in any pentagonal quasigroup.

Definition

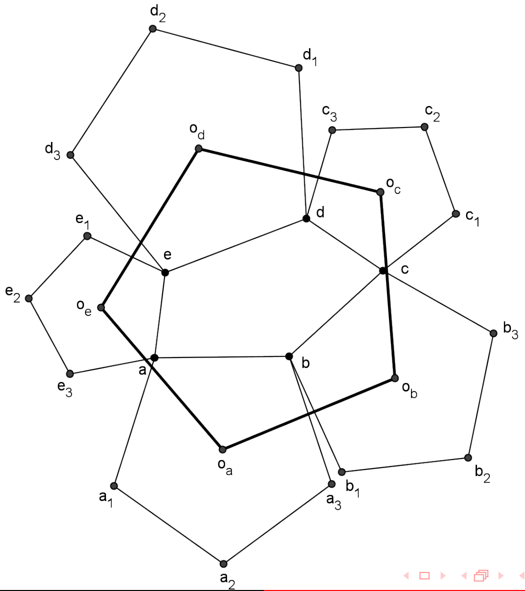
Let $(Q, \cdot)$ be a pentagonal quasigroup and a, b, c, d, $e \in Q$. We say that pentagon (a, b, c, d, e) is **affine regular pentagon**, *denoted by $ARP(a, b, c, d, e)$, if $d = (ca \cdot b)a \cdot c$ and $e = (ac \cdot b)c \cdot a$.*


ac
ca·b
d=(ca·b)a·c
e=(ac·b)c·a
c
a
b
(ca·b)a
ac·b
(ac·b)c
ca

Barlotti's theorem in pentagonal quasigroups:

Theorem

Let $(Q, \cdot)$ be a pentagonal quasigroup and $ARP(a, b, c, d, e)$, $RP(b, a, a_1, a_2, a_3)$ with center o_a, $RP(c, b, b_1, b_2, b_3)$ with center o_b, $RP(d, c, c_1, c_2, c_3)$, $RP(e, d, d_1, d_2, d_3)$ and $RP(a, e, e_1, e_2, e_3)$. If $RP(o_a, o_b, o_c, o_d, o_e)$, then o_c, o_d and o_e are centers of regular pentagons (d, c, c_1, c_2, c_3), (e, d, d_1, d_2, d_3) and (a, e, e_1, e_2, e_3), respectivelly.


d2
d1
c3
c2
od
d3
oc
d
c1
e1
e
c
e2
oe
b3
a
b
e3
ob
oa
b2
b1
a1
a3
a2

Contents

- develop more geometry of pentagonal quasigroups

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- determine the set of possible orders of finite pentagonal quasigroups

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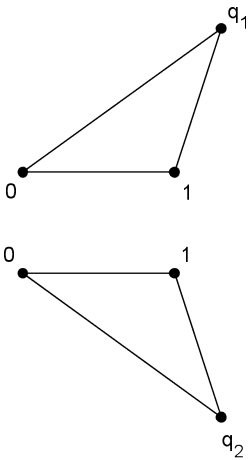
- plane tilings in pentagonal quasigroups


q1
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q3
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q2
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