Finitely generated varieties which are finitely decidable are residually finite

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The Finite Decidability Problem

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- *If* **A** *has any congruence covers of the lattice or semilattice types, or*
- *If any boolean- or affine-type minimal sets in* **A** *have nonempty tails, or*
- *If* **A** *is a subdirectly irreducible finite algebra with two incomparable nonabelian congruences,*

then every variety containing **A** *is finitely undecidable.*

Fact (continued)

If **A** *is a finite algebra*

- *and* **A** *has a solvable congruence which is nonabelian, or*

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These facts (and many of a similar nature) were established for modular varieties in the 90s (see [Idziak 1997]). The results for nonmodular varieties are in most cases new.

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Using familiar methods from the congruence-modular case, we show that

- every SI with boolean-type monolith belongs to $\mathrm{HS}(\mathcal{K})$;
- there is a bound ($\sim$ quadruply exponential) on the affine-type SIs.

Bounding unary-type SIs in $\mathcal{V}$

So let $\mathbf{S} \in \mathcal{V}$ have monolith $\perp \overset{1}{\prec} \mu$.

Lemma

$\mathrm{Rad}_u(\mathbf{S})$ *is comparable to all congruences of* $\mathbf{S}$.

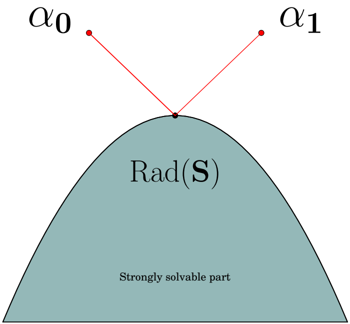
Lemma

$\mathrm{Rad}_u(\mathbf{S})$ *is meet-irreducible.*

Each of these is proved by contradiction: supposing the respective lemma were false, we construct a (relatively straightforward) interpretation of some finitely undecidable class into $\mathrm{HSP}(\mathbf{S})$.

Meet-irreducibility of the solvable radical

Goal: to semantically interprect a structure of the form $\langle I; E_0, E_1 \rangle$ (where the E_j are disjoint equivalence relations) into subpowers of $\mathbf{S}$.



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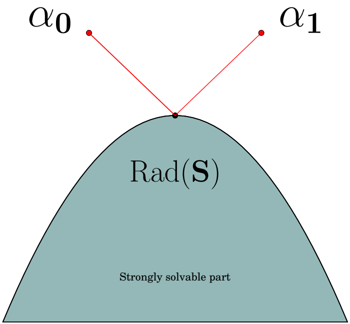

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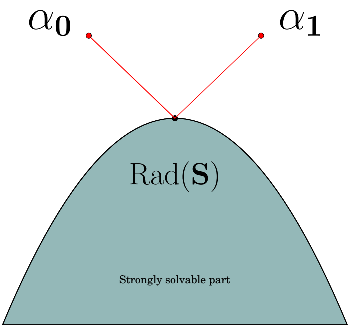
Using a failure of $\mathrm{C}(\mu, \{0_j, 1_j\}; \perp_S)$, and some tricks from tame congruence theory,



we reconstruct the original structure $\langle I; E_0, E_1\rangle$ in a first-order way from $\mathbf{B}$.

Since $\mathrm{Rad}_u(\mathbf{S})$ is meet-irreducible, we know that its index cannot exceed the maximum size of a boolean-type SI in $\mathcal{V}$.

Theorem

$\mathrm{Rad}_u(\mathbf{S})$ *is strongly abelian.*

Proof.

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Long! □

Takeaway idea: Subalgebra generation (and congruence generation) can frequently be proven to be "sparse" in some useful sense, when the generators are chosen so that they are almost constant modulo a strongly abelian congruence (such as the monolith).

Sparse subalgebra generation: Example I

Suppose $\mathrm{C}(\theta_0, \mu_{|U}; \perp)$ holds in our subdirectly irreducible algebra, but $\mathrm{C}(\theta_1, \mu_{|U}; \perp)$ does not, where $\theta_0 \prec \theta_1$ are strongly solvable.

See Lemma 3.1 in our paper for more about this example.

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Choose a witnessing package

$$t(a_0, \vec{b}_0) = t(a_0, \vec{b}_1)$$

but

$$t(a_1, \vec{b}_0) \neq t(a_1, \vec{b}_1)$$

where t takes values in some $\perp, \mu$-minimal set.

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Now suppose $\mathbb{G} = \langle V, E\rangle$ is a graph we want to interpret into $\mathrm{HSP}(\mathbf{S})$. Generate $\mathbf{D} \leq \mathbf{S}^{V \sqcup \{\infty\}}$ using all the diagonal elements, plus all elements of the form

$$a_{1|\{v,\infty\}} \oplus a_{0|\text{else}} \quad (v \in V)$$
$$a_{1|\{v,w,\infty\}} \oplus a_{0|\text{else}} \quad (v \overset{E}{—} w)$$

plus one extra element $m_{0|V} \oplus m_{1|\{\infty\}}$ (m_0, m_1 belonging to some $(\perp, \mu)$-trace).

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Key Claim

Every point in $D \cap U$ attains at most two values (mod θ_0), and does so precisely in the pattern of one of the generators (i.e. one of these values occurs at a vertex and infinity, or at the endpoints of an edge and at infinity).

Sparse subalgebra generation: Example II

Fix a $(\perp, \mu)$-minimal set U, and say we are working to semantically interpret a graph $\langle V, E\rangle$ into a power of $\mathbf{S}$. Let $I = \{v^+, v^- : v \in V\}$. Define a subalgebra

$$\Delta \subseteq \mathbf{B} \leq \mathbf{S}^I$$

with generators those $\mathbf{x} \in U^I$ such that for some $v \in V$,

$$\begin{cases} x^{v^+} \equiv_\mu x^{v^-} \\ x^{w^+} = x^{w^-} \equiv_\mu x^{v^+} \quad \text{for all other } w \in V \end{cases}$$

Example II continued

Claim

$B \cap U^I$ consists of just the generators and no more.

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Proof: write an arbitrary element $\mathbf{y} \in U^I$ as a product $\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of generators, where $\mathbf{f} = f^I$ for some polynomial operation $f : \mathbf{S} \to U$. Let C_j be the μ-class where $\mathbf{x}_j$ lives; then on $C_1 \times \cdots \times C_k$, f is essentially unary; say it depends on $\mathbf{x}_1$, which has its spike at $v_0 \in V$. Then $y^{v_0^+} \equiv_\mu y^{v_0^-}$, and for all $w \neq v_0$,

$$x_1^{w^+} = x_1^{w^-} \qquad \text{and} \qquad x_j^{w^+} \equiv_\mu x_j^{w^-}$$

so that

$$y^{w^+} = f(x_1^{w^+}, \ldots, x_k^{w^+}) = f(x_1^{w^-}, \ldots, x_k^{w^-}) = y^{w^-}$$

Sparse congruence generation: Example III

Assume that $\sigma = \mathrm{Rad}_u(\mathbf{S})$ is abelian over μ but not over $\bot$, and let $\mathbb{G} = \langle V, E \rangle$ be a graph. Fix the index set $I = V \times \{+, -\} \sqcup \{\infty\}$, and let $\mathbf{D} \leq \mathbf{S}^I$ be the subalgebra consisting of all σ-constant points.

See Lemma 3.6 in our paper for all the hypotheses of this example.

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Next, choose a $(\perp, \mu)$-subtrace $\{m_0, m_1\}$, and let Θ be the congruence on $\mathbf{D}$ generated by identifying all pairs

$$m_{1|v^+} \oplus m_{0|\text{else}} \equiv m_{1|v^-} \oplus m_{0|\text{else}} \qquad (v \in V)$$

$$m_{1|v^+,w^+} \oplus m_{0|\text{else}} \equiv m_{1|v^-,w^-} \oplus m_{0|\text{else}} \qquad (v \overset{E}{\text{---}} w)$$

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Key Claim

When restricted to a minimal set, Θ contains blocks of cardinality 1 and 2 only, and if $\mathbf{x} \equiv_\Theta \mathbf{y}$ then the set of coordinates where they differ is either empty, or $\{v^+, v^-\}$ for some $v \in V$, or $\{v^+, w^+, v^-, w^-\}$ for some $v \overset{E}{—} w$.

Bounding $\mathrm{Rad}_u(\mathbf{S})$-blocks

Say $\mathrm{Rad}_u(\mathbf{S})$ has index ℓ and some fixed monolith pair $c \neq d$.
Since $\mathrm{Rad}_u(\mathbf{S})$ is strongly abelian,

Lemma

For any polynomial $t(v_0, \vec{v}_1, \ldots, \vec{v}_\ell)$, there exist subsets of each variable set $\vec{v}_i$, of size no more than $\log |\mathbf{F}_{\mathcal{V}}(2+\ell)|$, such that for all $\mathrm{Rad}_u(\mathbf{S})$-blocks $B_1, \ldots, B_\ell$, the mapping

$$A \times \vec{B}_1 \times \cdots \times \vec{B}_\ell \to A$$

induced by t depends only on v_0 and the indicated subsets.

Because of the Lemma, terms $f(v_0) = t(v_0, \vec{s})$ of bounded arity suffice to send exactly one of any unequal elements $x_1 \neq x_2$ to c.

Bounding $\mathrm{Rad}_u(\mathbf{S})$-blocks, II

Consider a fixed $\mathrm{Rad}_u(\mathbf{S})$-block B, and to each $b \in B$ associate the set of terms $t(v_0, v_1, \ldots, v_k)$, with k bounded as described in the last slide, such that for some $p_1, \ldots, p_k$ from the appropriate $\mathrm{Rad}_u(\mathbf{S})$-blocks, $t(b, \vec{p}) = c$.

Claim

This is an injective map from B to subsets of $\mathbf{F}_{\mathcal{V}}(1 + k)$

For if not, we get a failure of the strong term condition

$$c = t(b_1, \vec{p}_1) = t(b_2, \vec{p}_2) \text{ but } t(b_2, \vec{p}_1) \neq c$$

This contradiction completes the proof.

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Problem

In all known cases, the set of finitely refutable sentences of a finitely generated variety is either decidable or Turing-complete. Do there exist varieties where this set has an intermediate complexity class?

Thank you!