Free idempotent generated semigroups

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All of old. Nothing else ever. Ever tried. Ever failed. No matter. Try again.

(S. Beckett, Worstword Ho)

Free IG semigroups: idea

- To every semigroup S with idempotents E associate the free-est semigroup $\mathrm{IG}(E)$ in which idempotents have the same structure as in S.
- To every regular semigroup S with idempotents E associate the free-est regular semigroup $\mathrm{RIG}(E)$ in which idempotents have the same structure as in S.
- Structure = biorder.

Free IG semigroups: definition

E – the set of idempotents in a semigroup S.

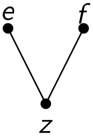
$$\mathrm{IG}(E) := \langle E \mid e^2 = e \quad (e \in E), \\ e \cdot f = ef \ (\{e,f\} \cap \{ef, fe\} \neq \emptyset)\rangle.$$

Suppose now S is regular.
$S(e,f) = \{h \in E \ : \ ehf = ef, \ fhe = h\} \neq \emptyset$ (sandwich sets).

$$\mathrm{RIG}(E) := \langle E \mid \mathrm{IG}, \ e \cdot h \cdot f = e \cdot f \ (e,f \in E, \ h \in S(e,f))\rangle.$$

Example: V-semilattice

Let $S =$



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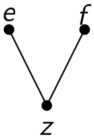
Let $S =$ 

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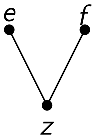

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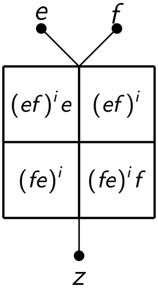

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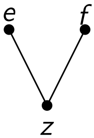
Let $S =$ 

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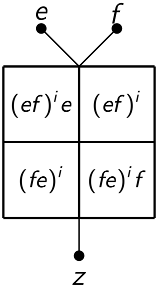

$RIG = \langle e, f, z \mid \mathrm{IG},\ ef = fe = z \rangle = S.$

Example: 2×2 rectangular band

$S = \langle e_{ij} \mid e_{ij} e_{kl} = e_{il} \ (i, j, k, l \in \{1, 2\}) \rangle$:

e_{11}	e_{12}
e_{21}	e_{22}

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$(e_{11}e_{22})^i e_{11}$ $(e_{12}e_{21})^i$	$(e_{12}e_{21})^i e_{12}$ $(e_{11}e_{22})^i$
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$\mathrm{RIG}(S) = \mathrm{IG}(S)$.

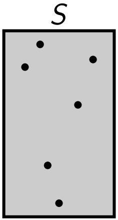
S, $\mathrm{IG}(E)$, $\mathrm{RIG}(E)$

- The sets of idempotents isomorphic (as biordered sets).
- The $\mathcal{D}$-class of an idempotent e has the same dimensions in all three.
- The group H_e in S is a homomorphic image of its counterparts in $\mathrm{IG}(E)$, $\mathrm{RIG}(E)$, which themselves are isomorphic.
- $\mathrm{IG}(E)$ may contain other, non-regular $\mathcal{D}$-classes.

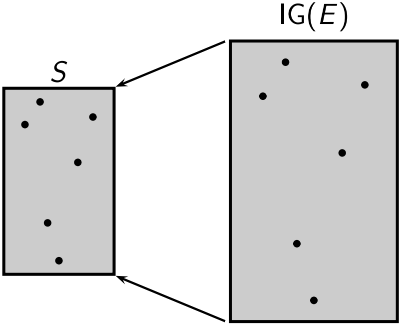
Question

Describe maximal subgroups of $\mathrm{IG}(E)$ and $\mathrm{RIG}(E)$.

Setting the problem: big picture



Setting the problem: big picture



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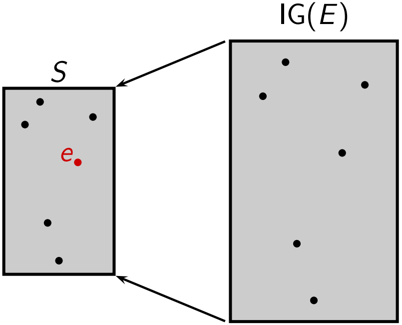

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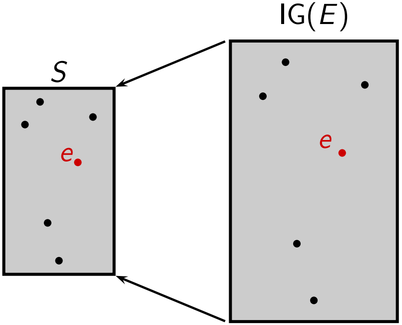

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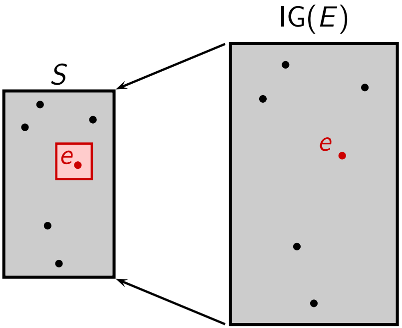

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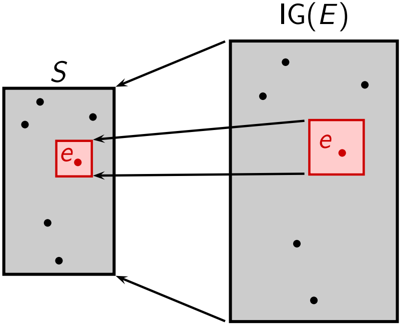

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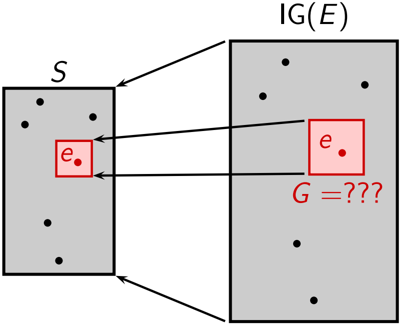
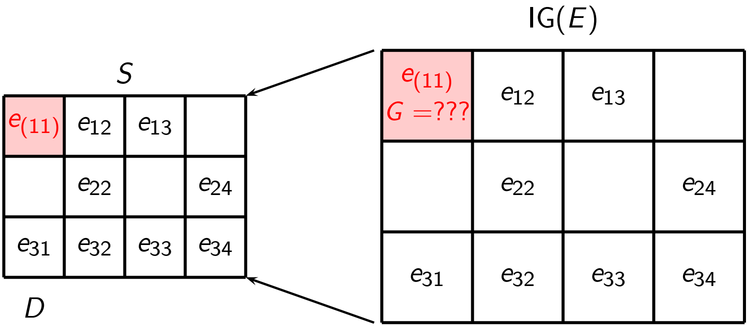


Setting the problem: zoom in



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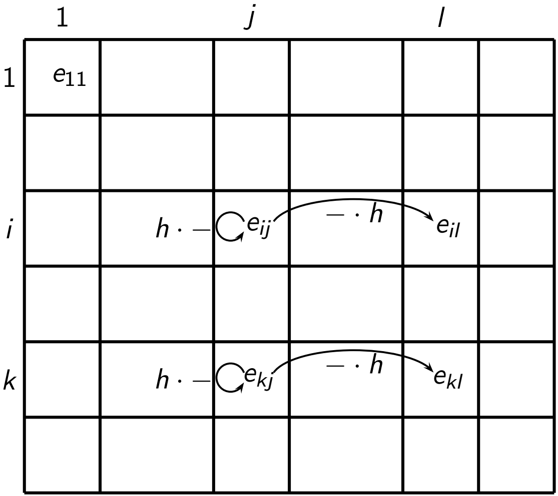
$$G = \langle f_{ij}\ (e_{ij} \in D \cap E) \mid ??? \rangle$$

Typical relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$

- $h = h^2$



Singular square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix}$; relation: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$.

Presentation

Theorem (Nambooripad '79; Gray, NR '12)

The maximal subgroup G of $e \in E$ in $IG(E)$ or $RIG(E)$ is defined by the presentation:

$$\begin{aligned}\langle f_{ij} \mid f_{i,\pi(i)} &= 1 && (i \in I),\\ f_{ij} &= f_{il} && (\text{if } r_j e_{il} \text{ is a Schreier rep}),\\ f_{ij}^{-1} f_{il} &= f_{kj}^{-1} f_{kl} && \left(\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \text{ sing. sq.}\right)\rangle.\end{aligned}$$

Remarks (1)

- Proof: Reidemeister–Schreier followed by Tietze transformations.
- Two types of relations:
  - Initial conditions: declaring some generators equal to 1 or each other;
  - Main relations: one per singular square.
- All relations of length ≤ 4.
- If no singular squares, the group is free.
- They have been conjectured to *always* be free.
- Brittenham, Margolis, Meakin '09 construct a 73-element semigroup such that $\mathrm{IG}(E)$ and $\mathrm{RIG}(E)$ have $\mathbb{Z} \times \mathbb{Z}$ as a maximal subgroup.

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- $\begin{bmatrix} a & a \\ b & c \end{bmatrix} \Rightarrow b = c$.
- $\begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix} \Rightarrow a = 1$.

Results (1): Gray, NR '12

Theorem

Every (finite) group is a maximal subgroup of some free regular idempotent generated semigroup (over a finite semigroup).

Theorem

Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.

Remark

Maximal subgroups of free idempotent generated semigroups arising from finite semigroups have to be finitely presented by Reidemeister–Schreier.

Remaining Question

Is every finitely presented group a maximal subgroup in some free idempotent generated semigroup over a finite regular semigroup?

Results (2): calculating the groups

Some or all maximal subgroups in $\mathrm{IG}(E(S))$ have been calculated for the following S:

- Full matrix monoid over a finite field: Brittenham, Margolis, Meakin; Dolinka, Gray.
- Full and partial transformation monoids: Gray, NR; Dolinka.
- Endomorphism monoid of a free G-act: Gould, Yang.

Results (3): bands

Theorem (Dolinka)

For every left- or right seminormal band B, all maximal subgroups of $IG(B)$ are free. For every variety $\mathbf{V}$ not contained in $\mathbf{LSNB} \cup \mathbf{RSNB}$ there exists $B \in \mathbf{V}$ such that $IG(B)$ contains a non-free maximal subgroup.

Remaining Question

Which subgroups arise as maximal subgroups of $\mathrm{IG}(B)$, B a band?

New construction (Dolinka, NR): set-up

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Let's obtain

$$\langle a, b, c \mid ab = c,\ bc = a,\ ca = b \rangle$$

as a maximal subgroup of $\mathrm{IG}(B)$ for a band B.

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Let's obtain

$$\langle a, b, c \mid ab = c,\ bc = a,\ ca = b \rangle (= Q_8 = F(2,3))$$

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- $I = \{0, a, b, c, 0', a', b', c'\}$;
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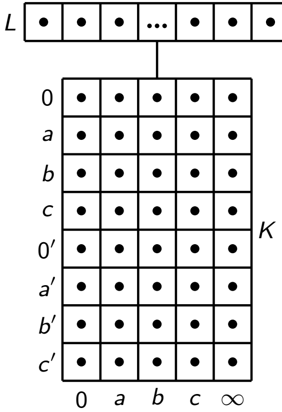
- $J = \{0, a, b, c, \infty\}$;
- $\mathcal{T} = \mathcal{T}_I^* \times \mathcal{T}_J$;
- the minimal ideal: $K = \{(\sigma, \tau) : \sigma, \tau \text{ constants}\}$;
- K is an $I \times J$ rectangular band.

New construction: set-up

- $B = K \cup L$, where L is a left zero semigroup.



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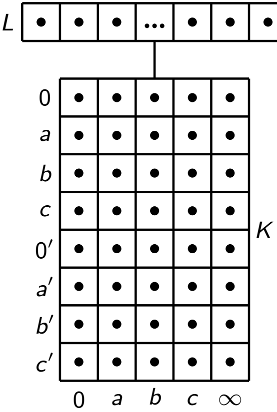

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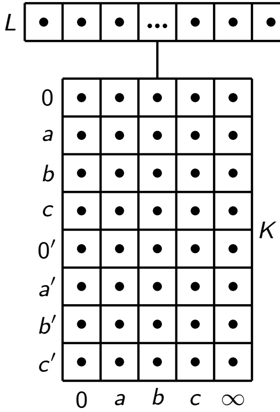

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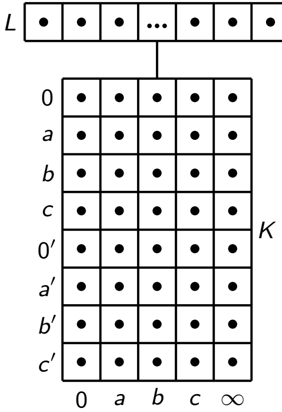

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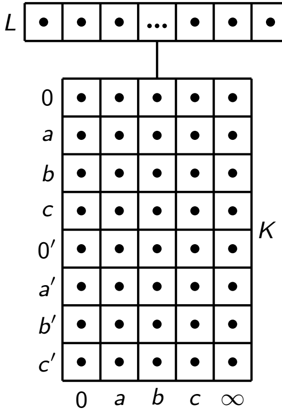

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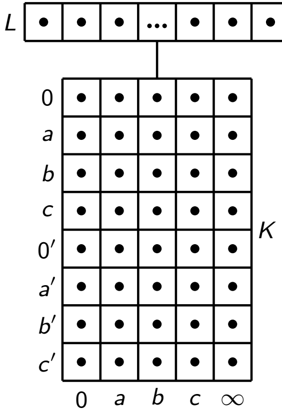

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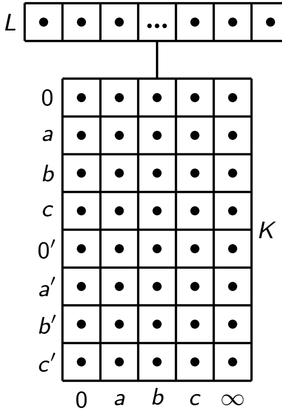
  - thus τ is specified by $(\infty)\tau$.



New construction: process

	0	a	b	c	∞
0	f_{00}	f_{0a}	f_{0b}	f_{0c}	$f_{0\infty}$
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	f_{c0}	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	$f_{0'0}$	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	$f_{a'0}$	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	$f_{b'0}$	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	$f_{c'0}$	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
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c'	$f_{c'0}$	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

Initial relations

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

Initial relations

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	f_{ab}	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	f_{ab}	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	f_{ac}	$f_{a\infty}$
b	1	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	1	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{0'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	1	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	1	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{a'\infty}$
b'	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{b'\infty}$
c'	1	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

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0	1	1	1	1	1
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c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	$f_{0'\infty}$
a'	1	a	b	c	$f_{a'\infty}$
b'	1	a	b	c	$f_{b'\infty}$
c'	1	a	b	c	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

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New construction: process

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a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	$f_{a'\infty}$
b'	1	a	b	c	$f_{b'\infty}$
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$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & 0' & 0' & 0' & 0' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	$f_{a'\infty}$
b'	1	a	b	c	$f_{b'\infty}$
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$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
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$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
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a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	$f_{b'\infty}$
c'	1	a	b	c	$f_{c'\infty}$

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ 0 & 0 & 0 & 0 & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
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c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	$f_{a\infty}$
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	$f_{b\infty}$
c	1	1	1	1	$f_{c\infty}$
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & 0 \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ b & b & b & b & c' & c' & c' & c' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ b & b & b & b & c' & c' & c' & c' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ b & b & b & b & c' & c' & c' & c' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & a \end{pmatrix}$$

$$G = \langle a, b, c \mid ab = c,$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ c & c & c & c & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & b \end{pmatrix}$$

$$G = \langle a, b, c \mid ab = c,$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ c & c & c & c & a' & a' & a' & a' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & b \end{pmatrix}$$

$$G = \langle a, b, c \mid ab = c,\ bc = a,$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & b' & b' & b' & b' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & c \end{pmatrix}$$

$$G = \langle a, b, c \mid ab = c,\ bc = a,$$

New construction: process

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$\sigma = \begin{pmatrix} 0 & a & b & c & 0' & a' & b' & c' \\ a & a & a & a & b' & b' & b' & b' \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & a & b & c & \infty \\ 0 & a & b & c & c \end{pmatrix}$$

$$G = \langle a, b, c \mid ab = c,\ bc = a,\ ca = b \rangle$$

Subgroups of IG(B), B band

Theorem

For any group G there exists a band B such that $IG(B)$ has a maximal subgroup isomorphic to G. Furthermore, if G is finitely presented, then B can be constructed to be finite.

Remark

- G has two $\mathcal{D}$-classes, K and L.
- If $G = \langle A \mid R \rangle$ with $|A| = m$, $|R| = n$, then
  - K is a $(2m + 2) \times (m + 2)$ rectangular band;
  - L is a left zero semigroup of order $2m + n + 1$.

Future directions: word problem

Fact

Suppose S is a finite regular semigroup. The word problem for $RIG(E)$ is solvable iff the word problem for each of its maximal subgroups is solvable.

Open Problem

Is it true that the word problem for $\mathrm{IG}(E)$ arising from a finite semigroup is solvable iff the word problem for each of its maximal subgroups is solvable?

Open Problem

Is it true that the word problem for $\mathrm{IG}(E)$ arising from a finite semigroup is solvable iff the word problem for each of its Schützenberger groups is solvable?

Open Problem

Can $\mathrm{IG}(E)$ have non-trivial Schützenberger groups in non-regular $\mathcal{D}$-classes?

... Try again. Fail again. Fail better.

(S. Beckett, Worstword Ho)