# ON THE GENUS OF THE INTERSECTION GRAPH OF IDEALS OF A COMMUTATIVE RING

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**Definition (Chakrabarty, Ghosh, Mukherjee, Sen)**

Let $R$ be a commutative ring and $I^*(R)$ the set of its nontrivial ideals. *The intersection graph of ideals* $G(R)$ is defined as follows:

$$V(G(R)) := I^*(R), \quad E(G(R)) := \{\{I_1, I_2\} : I_1 \cap I_2 \neq 0\},$$

where $V(G(R))$ (resp. $E(G(R))$) denotes the set of vertices (resp. edges) of the graph $G(R)$.

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- (Thomassen) Determining the genus of a graph is NP-complete problem.

## Planarity and toroidality of intersection graphs

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**Theorem (Erić, Pucanović, Radovanović)**

Let $R$ be a commutative ring with identity. Graphs $G(R)$ with $\gamma(G(R)) = 2$ are $\Gamma'$ and some subgraph of $\Gamma''$.

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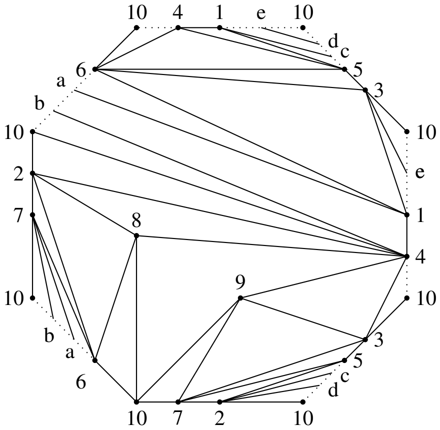


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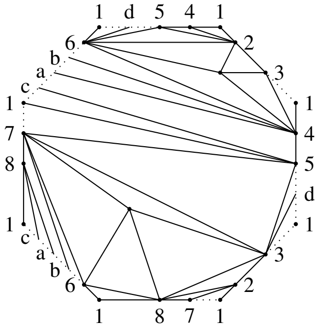
  - $|R_1/M_1| = 2 \Rightarrow G(R)$ is isomorphic to the graph in the picture.



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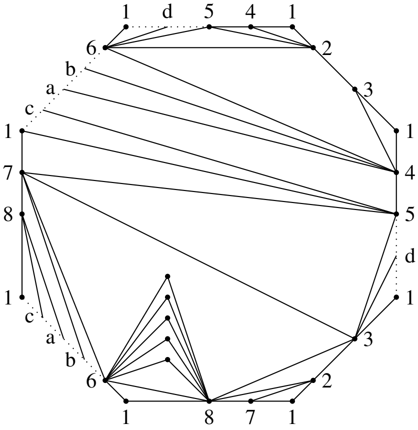
- $|R/M| = 5$

## Sketch of the proof (part 4)

Finally, let $R$ be a local ring with maximal ideal $M$ which is minimally generated with two element. If $M^2 = \langle u^2 \rangle$, $uv = 0$, $u^2 = 0$, then:

- $|R/M| \leqslant 4 \Rightarrow G(R)$ is toroidal.
- $|R/M| \geqslant 6 \Rightarrow G(R)$ contains $K^9$.
- $|R/M| = 5 \Rightarrow G(R)$ is isomorphic to the graph in the picture.



# Genus of the intersection graph of nonlocal rings

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- Let $\alpha_i = \omega(G(R_i))/|V(G(R_i))|$ if $R_i$ is not a field, and $\alpha_i = 3/2$ otherwise. Then,

$$\omega(G(R)) \geqslant \max\{\alpha_i \mid 1 \leqslant i \leqslant n\} \cdot \frac{N}{3},$$

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**Theorem (Erić, Pucanović, Radovanović)**

Genus of the intersection graph of a nonlocal ring $R$ is at least

$$\min\left\{\frac{\alpha}{8} \cdot N^{\frac{2k-2}{k}} \cdot (N^{1/k} - \alpha) - \frac{N}{2} + 1, \beta \cdot N^2 - \frac{N}{2} + 1, \frac{(N-6)(N-8)}{48}\right\},$$

where $N = |V(G(R))|$, $\alpha = 2k\left(\frac{1}{3}\right)^{\frac{k-1}{k}}$ and $\beta = \frac{3^k - 2^k - 1}{4 \cdot (2 \cdot 3^k - 2^{k+1} - 1)^2}$.

# Number of genus $g$ intersection graphs

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**Theorem (Erić, Pucanović, Radovanović)**

For every $g > 0$, there are only finitely many nonisomorphic graphs of genus $g$ that are intersection graphs of some rings.

The end.
Thank you for your attention.