Congruence lattices and Compact Intersection Property

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Congruence lattices

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
(Con $\mathcal{K} \subseteq$ Con $\mathcal{L}$)

Satisfactory description

Some varieties $\mathcal{V}$ for which Con $\mathcal{V}$ is well understood:

- Boolean algebras (bounded distributive lattices);
- distributive lattices;
- Stone algebras.

What do they have in common?

- congruence-distributive;
- finitely generated;
- Compact Intersection Property: intersection of compact (finitely generated) congruences is always compact.

Why is CIP so helpful?

Compact congruences of an algebra A form a $(\vee, 0)$-subsemilattice $\mathrm{Con}_c\, A$ of the lattice $\mathrm{Con}\, A$. The lattice $\mathrm{Con}\, A$ is isomorphic to the ideal lattice of $\mathrm{Con}_c\, A$. And if the semilattices $\mathrm{Con}_c\, A$ for $A \in \mathcal{V}$ fail to be lattices, then they are difficult to characterize, because various "refinement properties" come into play.

Compare

Theorem

(F. Wehrung) There exists a distributive algebraic lattice which is not isomorphic to the congruence lattice of any lattice.

with

Theorem

(E. T. Schmidt) Every distributive algebraic lattice whose compact elements are closed under intersection is isomorphic to the congruence lattice of a lattice.

Problem

So, we consider the following general problem.

Given a finitely generated CD variety $\mathcal{V}$ with CIP, characterize lattices in $\mathrm{Con}_c\,\mathcal{V}$.

CIP varieties

Theorem

For a finitely generated congruence-distributive variety $\mathcal{V}$, the following conditions are equivalent

1. $\mathcal{V}$ *has CIP.*
2. *Every subalgebra of a subdirectly irreducible algebra is subdirectly irreducible or one-element;*
3. *For every embedding $f: A \to B$ with A finite the mapping $\mathrm{Con}_c f: \mathrm{Con}_c A \to \mathrm{Con}_c B$ preserves meets.*

($\mathrm{Con}_c f(\alpha)$ is the congruence on B generated by all pairs $(f(x), f(y))$ with $(x, y) \in \alpha$. The map $\mathrm{Con}_c f$ always preserves 0 and joins.)
(Equivalence of (1) and (2) observed by Baker, proved by Blok, Pigozzi.)

Irreducibles

$\mathcal{V}$.... a finitely generated, congruence-distributive CIP variety (throughout the talk).

$\mathrm{SI}(\mathcal{V})$.....all subdirectly irreducible members of $\mathcal{V}$, including one-element algebras;

$\mathrm{M}^*(L)$.......all completely $\wedge$-irreducible elements of a lattice L, including 1.

Obvious: $\alpha \in \mathrm{M}^*(\mathrm{Con}\, A)$ iff $A/\alpha \in \mathrm{SI}(\mathcal{V})$.

Valuations

Let P be a poset with 1. *A* $\mathrm{SI}(\mathcal{V})$*-valuation* is a P-indexed commutative diagram $(v(p), f_{p,q}; p, q \in P,\ p \le q)$ such that

- $v(p) \in \mathrm{SI}(\mathcal{V})$ for every p;
- $f_{p,q}$ is a surjective homomorphism for every $p \le q$;
- the assingment $q \mapsto \ker(f_{p,q})$ is a bijection from $\uparrow p$ to $M^*(\mathrm{Con}\, v(p))$ (in fact, an order-isomorphism)

Example: If A is any algebra, $P = M^*(\mathrm{Con}\, A)$, and $f_{\alpha\beta}$ is the natural projection $A/\alpha \to A/\beta$, then $(A/\alpha, f_{\alpha,\beta})$ is a $\mathrm{SI}(\mathcal{V})$-valuation.

Finite level criterion

Theorem

For a finite lattice L, the following are equivalent.

(1) $L \in \mathrm{Con}\,\mathcal{V}$;

(2) *there exists a* $\mathrm{SI}(\mathcal{V})$*-valuation* $D = (v(p), f_{p,q})$ *on* $P = \mathrm{M}^*(L)$ *such that*

  (i) *all projections* $\pi_p : \lim_{\leftarrow} D \to v(p)$ *are surjective;*

  (ii) *if* $p \nleq q$, *then* $\ker(\pi_p) \nsubseteq \ker(\pi_q)$.

Valuations satisfying (i) and (ii) will be called *admissible*.

Duals of lattice homomorphisms

Now let $\varphi : K \to L$ be a $(0, \vee)$-homomorphism of finite $(0, \vee)$-semilattices. We define the map $\varphi^{\leftarrow} : L \to K$ by

$$\varphi^{\leftarrow}(\beta) = \bigvee\{\alpha \mid \varphi(\alpha) \leq \beta\}.$$

If $K = \operatorname{Con} A$, $L = \operatorname{Con} B$ and $\varphi = \operatorname{Con} f$, for some algebras A, B and a homomorphism $f : A \to B$, then $\varphi^{\leftarrow}(\beta) = \{(x, y) \in A \mid (f(x), f(y)) \in \beta\}$.

Lemma

Let $\varphi : K \to L$ be a $(0, \vee)$-homomorphism of finite lattices.

1. $\varphi^{\leftarrow}$ *preserves* $\wedge$ *and* 1.
2. $\varphi(\alpha) = \bigwedge\{\beta \mid \alpha \leq \varphi^{\leftarrow}(\beta)\}$.
3. *If $\varphi : K \to L$ is a 0-preserving homomorphism of finite distributive lattices, then $\varphi^{\leftarrow}(c) \in \mathrm{M}^*(K)$ for every $c \in \mathrm{M}^*(L)$.*

Con $\mathcal{V}$ via direct limits

Theorem

Let L be a distributive lattice with 0. Then $L \simeq \mathrm{Con}_c A$ for some $A \in \mathcal{V}$ if and only if L is isomorphic to the direct limit of a P-indexed diagram $\vec{L} = (L_p, \varphi_{p,q} \mid p \leq q \text{ in } P)$, where each L_p is a finite distributive lattice and each $\varphi_{p,q}$ is a 0-preserving lattice homomorphism, such that

- *For every $p \in P$, the ordered set $\mathrm{M}^*(L_p)$ has an admissible $\mathrm{SI}(\mathcal{V})$-valuation $(v_p(\alpha), f^p_{\alpha,\beta})$.*
- *For every $p, q \in P$, $p \leq q$ and for every $\alpha \in \mathrm{M}^*(L_q)$ there exists embedding*

$$e^{\alpha}_{p,q} : v_p(\varphi^{\leftarrow}_{p,q}(\alpha)) \to v_q(\alpha) \text{ such that}$$

$$e^{\beta}_{p,q} f^{p}_{\alpha',\beta'} = f^{q}_{\alpha,\beta} e^{\alpha}_{p,q},$$

for every $\alpha \leq \beta$ in $\mathrm{M}^*(L_q)$ and $\alpha' := \varphi^{\leftarrow}_{p,q}(\alpha), \beta' := \varphi^{\leftarrow}_{p,q}(\beta)$.

Example

Additional assumptions:

- for every $S \in \mathrm{SI}(\mathcal{V})$, either $|\operatorname{Con} S| = 1$ or $|\operatorname{Con} S| = 2$;
- there exists $S \in \mathrm{SI}(\mathcal{V})$ which has a one-element subalgebra.

Theorem

Let L be a distributive lattice with 0. TFAE

(1) $L \simeq \operatorname{Con}_c A$ *for some* $A \in \mathcal{V}$;

(2) L *is isomorphic to the direct limit of a* P*-indexed diagram* $\vec{L} = (L_p, \varphi_{p,q} \mid p \leq q \text{ in } P)$, *where each* L_p *is a finite boolean lattice and each* $\varphi_{p,q}$ *is a* 0*-preserving lattice homomorphism;*

(3) L *is a generalized Boolean lattice.*

$\mathrm{Con}\,\mathcal{V}$ via duality

Theorem

Let L be a distributive lattice with 0 and let $(P(L), \tau, \leq)$ be its dual Priestley space. The following are equivalent.

(1) $L \in \mathrm{Con}_c\,\mathcal{V}$;

(2) *There exists a $\mathrm{SI}(\mathcal{V})$-valuation $D = (v(I), f_{I,J})$ on $P(L)$ and a subalgebra A of $\lim_{\leftarrow} D$ such that*

   (i) *every projection $\pi_I : A \to v(I)$ is surjective;*

   (ii) *if $I \nleq J$, then $\ker(\pi_I) \nsubseteq \ker(\pi_J)$;*

   (iii) *for every $a, b \in A$ the set $U_{a,b} = \{I \mid \pi_I(a) = \pi_I(b)\}$ is clopen.*

Weaker conditions

Theorem

If $L \simeq \mathrm{Con}_c\, A$ *for some* $A \in \mathcal{V}$*, then*

(Pr1) $P(L)$ *has an admissible* $\mathrm{SI}(\mathcal{V})$*-valuation* $(v(I), f_{I,J})$*;*

(Pr2) *For every* $I \in P(L)$ *there exists an open set* U *such that* $I \in U$ *and for every* $J \in U$ *the algebra* $v(I)$ *is isomorphic to a subalgebra of* $v(J)$*.*

In many cases the conditions (Pr1) and (Pr2) are sufficient.

Example

Additional assumptions:

- for every $S \in \mathrm{SI}(\mathcal{V})$, $\mathrm{Con}\, S$ is a chain with $|\mathrm{Con}\, S| \leq n$
- if $S \leq T \in \mathrm{SI}(\mathcal{V})$, then $\mathrm{Con}\, S \simeq \mathrm{Con}\, T$.

Denotation:

$\mathcal{P}_n$...... the class of all partially ordered sets $(C, \leq)$ with a largest element, such that for every $x \in C$, $\uparrow x$ is a k-element chain, $k \leq n$

Example

Theorem

Let $\mathcal{V}$ satisfy the assumptions stated above. Let L be a distributive lattice with 0 and let $(P(L), \leq, \tau)$ be its dual Priestley space. The following conditions are equivalent.

(1) $L \simeq \mathrm{Con}_c A$ *for some* $A \in \mathcal{V}$;

(2) $(P(L), \leq, \tau)$ *satisfies* $(Pr1)$ *and* $(Pr2)$;

(3) $P(L) \in \mathcal{P}_n$ *and for every* $k = 1, \ldots, n$ *the set* $P_k(L) = \{x \in P(L) \mid |{\uparrow}x| = k\}$ *is clopen.*

(4) L *is a dual Stone lattice of order* n.

Conjecture: (Pr1) and (Pr2) are sufficient whenever $\mathrm{Con}\, S$ is a chain for every $S \in \mathrm{SI}(\mathcal{V})$.

(Pr1),(Pr2) not sufficient

$\mathcal{W}$ =HSP(A),
where
A is the 4-element chain $0 < a < b < 1$, regarded as a lattice, endowed with an additional unary operation
$h(0) = 0$, $h(a) = b$, $h(b) = a$, $h(1) = 1$. then:

- Every countable lattice satisfying (Pr1), (Pr2) belongs to $\mathrm{Con}_c\,\mathcal{W}$;
- There exist a lattice satisfying (Pr1), (Pr2) of cardinality $\aleph_1$ not in $\mathrm{Con}_c\,\mathcal{W}$.

Critical points

$\mathcal{V}$, $\mathcal{W}$.............classes of algebras

Gillibert:

$\mathrm{Crit}(\mathcal{V};\mathcal{W}) = \min\{|S| : S \in \mathrm{Con}_c\,\mathcal{V} \setminus \mathrm{Con}_c\,\mathcal{W}\}$
(∞ if $\mathrm{Con}_c\,\mathcal{V} \subseteq \mathrm{Con}_c\,\mathcal{W}$)

Theorem

There are finitely generated congruece distributive CIP varieties $\mathcal{V}$*,* $\mathcal{W}$) *with* $\mathrm{Crit}(\mathcal{V};\mathcal{W}) = \aleph_1$)*.*

Conjecture: always $\mathrm{Crit}(\mathcal{V};\mathcal{W}) \leq \aleph_1$ for such varieties.