On generating sets of polymorphism clones of homogeneous structures

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07.06.2013

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On clones ...

Given a set A.

- $O_A^{(n)} := \{f \mid f : A^n \to A\}$, $O_A := \cup_{n\in\mathbb{N}\setminus\{0\}} O_A^{(n)}$,
- $F^{(n)} := F \cap O_A^{(n)}$, for $F \subseteq O_A$,
- **Projections**: $J_A := \{e_i^n \mid e_i^n(x_1,\dots,x_n) = x_i, n \in \mathbb{N}\}$,
- **Composition**: For $f \in O_A^{(n)}$, $g_1,\dots,g_n \in O_A^{(m)}$

$$f \circ \langle g_1,\dots,g_n\rangle(x_1,\dots,x_m) := f(g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m))$$

$C \subseteq O_A$ is a **clone** on A if

- $J_A \subseteq C$, and
- whenever $f \in C^{(n)}$, $g_1,\dots g_n \in C^{(m)}$, then $f \circ \langle g_1,\dots,g_n\rangle \in C^{(m)}$.

. . . and their generating systems

- If $M \subseteq O_A$, then $\langle M \rangle_{O_A}$ is the smallest clone on A that contains M.
- If $C \leq A$ and $C = \langle M \rangle_{O_A}$, then M is a **generating system** for C.

Question

If we consider a structure $\mathbf{A}$ and its polymorphism clone $\operatorname{Pol}\mathbf{A}$, what can be said about its generating systems?
In particular, what happens if $\mathbf{A}$ is a homogeneous structure?

The evergreen result of Sierpiński

Theorem (Sierpiński (1945))

For an arbitrary set A holds

$$\left\langle O_A^{(2)} \right\rangle_{O_A} = O_A,$$

i.e. the clone of all operations on A is generated by its binary part.

Generating semigroups

- Ruškuc introduced in 1994 the notion of relative ranks.
- Higgins, Howie and Ruškuc showed in 1998 that the semigroup of all transformations on an infinite set A is generated by the set of permutations of A and two additional functions, i.e.

The semigroup of transformations of A has relative rank 2 modulo the full symmetric group on A.

Relative rank for clones

Let F be a clone on a set A and let $M \subseteq F$ be an arbitrary subset of F.

- A subset N of F is called **generating set of F modulo M** if

$$\langle M \cup N \rangle_{O_A} = F.$$

- The **relative rank** of F modulo M is the smallest cardinal of a generating set N of F modulo M, and is denoted by

$$\mathrm{rank}(F : M).$$

Beyond Sierpiński's theorem

Proposition

Let $\mathbf{A}$ *be a structure such that there exists a retraction* $r : \mathbf{A} \twoheadrightarrow \mathbf{A}^2$. *Then*

$$\operatorname{rank}(\operatorname{Pol}\mathbf{A} : \operatorname{End}\mathbf{A}) = 1.$$

In particular, $\operatorname{Pol}\mathbf{A}$ *is generated by* $\operatorname{End}\mathbf{A}$ *together with a section*

$$\epsilon : \mathbf{A}^2 \hookrightarrow \mathbf{A} \text{ with } r \circ \epsilon = 1_{\mathbf{A}^2}.$$

Example: Rado graph I

The **Rado-graph** is, up to isomorphism, the unique countably infinite graph R such that for all disjoint finite sets U, V of vertices there exists a vertex c joint to all elements of U and to none in V.

Theorem (Bonato, Delić 2000)

A countable graph G is isomorphic to a retract of the Rado graph if and only if G is algebraically closed.

Remark

A countable graph G is algebraically closed if every finite set $S \subseteq V(G)$ has a common neighbor.

Example: Rado graph II

Observation

The square R^2 of the Rado graph R is algebraically closed.

- Let $A \subseteq V(R^2)$, $A = \{(a_1, b_1), \dots, (a_n, b_n)\}$.
- For $U := \{a_1, \dots, a_n, b_1, \dots, b_n\}$ and $V := \emptyset$ exists a $c \in V(R)$ connected to all vertices of U.
- Consider $(c, c) \in V(R^2)$. This is a common neighbor of A.

Homogeneity . . .

Given is a structure **A**.

- A **local isomorphism** of a structure **A** is an isomorphism between finite substructures of **A**.
- A structure **A** is **homogeneous** if every local isomorphism of **A** extends to an automorphism of **A**.

... and AP (Amalgamation property)

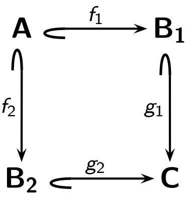
Let $\mathcal{C}$ be a class of structures. If

- $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$, and
- $f_1 : \mathbf{A} \hookrightarrow \mathbf{B_1}$ and $f_2 : \mathbf{A} \hookrightarrow \mathbf{B_2}$ are embeddings,

then there are

- $\mathbf{C} \in \mathcal{C}$, and
- embeddings $g_1 : \mathbf{B_1} \hookrightarrow \mathbf{C}$ and $g_2 : \mathbf{B_2} \hookrightarrow \mathcal{C}$

such that the following diagram commutes:



i.e.

$$g_1 \circ f_1 = g_2 \circ f_2.$$

Age

- The **age** of a structure $\mathbf{A}$ is the class of all finitely generated structures that embed into $\mathbf{A}$.

Let $\mathcal{C}$ be a class of finitely generated structures over the same signature.

- **Hereditary property** (HP)
  If $\mathbf{A} \in \mathcal{C}$, and $\mathbf{B} \hookrightarrow \mathbf{A}$, then $\mathbf{B} \in \mathcal{C}$.
- **Joint embedding property** (JEP)
  If $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, then there exists a $\mathbf{C} \in \mathcal{C}$ such that both $\mathbf{A}$ and $\mathbf{B}$ are embeddable in $\mathbf{C}$.

Theorem (Fraïssé)

$\mathcal{C}$ is the age of a countable structure iff it has, up to isomorphism, countably many structures, and it has the HP and the JEP.

Fraïssé-classes and Fraïssé-limit

- An age that has AP is called **Fraïssé class**.

Theorem (Fraïssé)

$\mathcal{C}$ is a Fraïssé class iff there is a countable homogeneous structure $\mathbf{U}$, such that $\mathcal{C}$ is the age of $\mathbf{U}$.
All countable homogeneous structures of age $\mathcal{C}$ are mutually isomorphic.

- $\mathbf{U}$ is called the **Fraïssé-limit** of the class $\mathcal{C}$.

Homomorphism-homogeneity. . .

Given is a structure **A**.

- A **local homomorphism** of a structure **A** is a homomorphism from a finite substructure of **A** to **A**.
- *Cameron and Nešetřil (2002)*:
  A structure **A** is **homomorphism-homogeneous** if every local homomorphism of **A** extends to an endomorphism of **A**.

. . . and HAP (Homo-almagamation property)

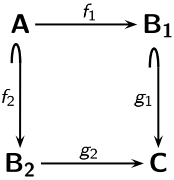
Let $\mathcal{C}$ be a class of structures. If

- $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{C}$,
- $f_1 : \mathbf{A} \to \mathbf{B_1}$ is a homomorphism, and
- $f_2 : \mathbf{A} \hookrightarrow \mathbf{B_2}$ is an embedding,

then there are

- $\mathbf{C} \in \mathcal{C}$,
- an embedding $g_1 : \mathbf{B_1} \hookrightarrow \mathbf{C}$, and
- a homomorphism $g_2 : \mathbf{B_2} \to \mathbf{C}$

such that the following diagram commutes:



i.e.

$$g_1 \circ f_1 = g_2 \circ f_2.$$

Amalgamated extension property (Kubiś)

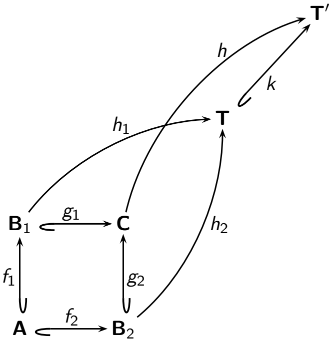
Let $\mathcal{C}$ be a class of countable, finitely generated structures. If

- $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{T} \in \mathcal{C}$,
- $f_1 : \mathbf{A} \hookrightarrow \mathbf{B_1}$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B_2}$ are embeddings, and
- $h_1 : \mathbf{B_1} \to \mathbf{T}$, $h_2 : \mathbf{B_2} \to \mathbf{T}$ are homomorphisms, with $h_1 \circ f_1 = h_2 \circ f_2$.

then there are

- $\mathbf{C}, \mathbf{T}' \in \mathcal{C}$,
- embeddings $g_1 : \mathbf{B_1} \hookrightarrow \mathbf{C}$, $g_2 : \mathbf{B_2} \hookrightarrow \mathbf{C}$, $k : \mathbf{T} \hookrightarrow \mathbf{T}'$ and
- a homomorphism $h : \mathbf{C} \to \mathbf{T}'$

such that the following diagram commutes:



Main result

Let $\operatorname{Emb}\mathbf{A}$ be the submonoid of $\operatorname{End}\mathbf{A}$ that consists of all homomorphic self-embeddings of $\mathbf{A}$.

Theorem

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$, such that

(1) *$\mathcal{C}$ is closed with respect to finite products;*

(2) *$\mathcal{C}$ has the HAP, and*

(3) *$\mathcal{C}$ has the amalgamated extension property.*

Then

$$\operatorname{rank}(\operatorname{Pol}\mathbf{A} : \operatorname{Emb}\mathbf{A}) \leq 2.$$

In particular, $\operatorname{Pol}\mathbf{U}$ *is generated by* $\operatorname{Emb}\mathbf{U}$ *together with an unary and a binary polymorphism.*

Further examples

The polymorphism clones of the following structures have relative rank at most 2 modulo the respective self-embedding monoids:

- the Rado graph R;
- the countable generic poset $\mathbb{P} = (P, \leq)$;
- the countable atomless Boolean algebra $\mathbb{B}$;
- the countable universal homogeneous lattice Ω;
- the countable universal homogeneous distributive lattice $\mathbb{D}$;
- the infinite-dimensional vector-space $\mathbb{F}^\omega$ for any countable field $\mathbb{F}$;
- the rational Urysohn space $\mathbb{U}_\mathbb{Q}$;
- the rational Urysohn sphere of radius 1.

An open problem

The age of $(\mathbb{Q}, \leq)$ is not closed with respect to finite products.

(1) Does $\mathrm{Pol}(\mathbb{Q}, \leq)$ have a generating set of bounded arity?

(2) What is its relative rank with respect to

$$\mathrm{End}(\mathbb{Q}, \leq),\ \mathrm{Emb}(\mathbb{Q}, \leq),\ \text{or even } \mathrm{Aut}(\mathbb{Q}, \leq)?$$