A Characterization of 2-supernilpotent Mal’cev Algebras

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Mal’cev Algebras

Definition

Mal’cev term: $d(x,y,y) = d(y,y,x) = x$

Expanded groups

An algebra $(V,+,-,0,F)$ is called an expanded group if $(V,+,-,0)$ is a group and F is a set of operations on V.

Examples of Mal’cev algebras

Groups, rings, modules, expanded groups, quasigroups,. . .

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Absorbing Polynomials

Definition

Let **A** be an algebra and let $n \in \mathbb{N}$, $(a_1, \ldots, a_n) \in A^n$, $a \in A$. An n-ary polynomial p is absorbing at $(a_1, \ldots, a_n)$ with value a if $p(x_1, \ldots, x_n) = a$ whenever there exists an $i \in \{1, \ldots, n\}$ such that $x_i = a_i$.

Absorbing polynomials in expanded groups

Let $n \in \mathbb{N}$. An n-ary polynomial f of an expanded group $(V, +, -, 0, F)$ is absorbing if $f(a_1, \ldots, a_n) = 0$ whenever there exists an $i \in \{1, \ldots, n\}$ such that $a_i = 0$.

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Absorbing Polynomials of a Small Arity

Definition

Let **A** be an algebra and let $a, b, c \in A^3$, $d \in A$. A ternary polynomial p is absorbing at (a, b, c) with value d if $p(x, y, z) = d$ whenever $x = a$ or $y = b$ or $y = c$.

Ternary absorbing polynomials in expanded groups

A ternary polynomial f of an expanded group $(V, +, -, 0, F)$ is absorbing if $f(x, y, z) = 0$ whenever $x = 0$ or $y = 0$ or $z = 0$.

Commutator polynomials in expanded groups

A binary absorbing polynomial f of an expanded group $(V, +, -, 0, F)$ is commutator polynomial.

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Algebras **A** and **B** are polynomially equivalent if Pol **A** = Pol **B**.

Theorem (R. Freese, R.N. McKenzie)

Every nilpotent Mal'cev algebra is polynomially equivalent to an expanded loop.

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Commutators

In Groups

If H, K are normal subgroups of a group $\mathbf{G}$ then $[H, K]$ is a normal subgroup generated by $\{[h,k] \mid h \in H,\ k \in K\}$, where $[h,k] := h^{-1}k^{-1}hk$ for all $h \in H$ and $k \in K$.

TC commutator (R. Freese, R.N. McKenzie)

The term condition commutator $[\bullet, \bullet]$ in a Mal'cev algebra $\mathbf{A}$ is a binary operation on Con $\mathbf{A}$, defined by the centralizing relation.

Proposition (E. Aichinger, N. M., 2010)

The binary commutator $[1, 1]$ of a Mal'cev algebra $\mathbf{A}$ is the congruence of $\mathbf{A}$ generated by $\{(p(a_1, b_1), p(a_2, b_2)) \mid a_1, a_2, b_1, b_2 \in A, p$ is absorbing at (a_2, b_2) with value $p(a_2, b_2)\}$.

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Higher Commutators

A. Bulatov, 2001

The term condition n-ary commutator $[\underbrace{\bullet, \dots, \bullet}_{n}]$ in a Mal'cev algebra is an n-ary operation on $\operatorname{Con} \mathbf{A}$, defined by the higher centralizing relation.

Special case (E. Aichinger, N. M., 2010)

The ternary commutator $[1, 1, 1]$ of a Mal'cev algebra $\mathbf{A}$ is the congruence of $\mathbf{A}$ generated by $\{(p(a_1, b_1, c_1), p(a_2, b_2, c_2)) \mid a_1, a_2, b_1, b_2, c_1, c_2 \in A, p$ is absorbing at (a_2, b_2, c_2) with value $p(a_2, b_2, c_2)\}$.

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Abelian, Nilpotent and Supernilpotent

Abelian

Algebras that satisfy $[1,1]=0$ are called abelian.

2-nilpotent (R. Freese, R.N. McKenzie)

Algebras that satisfy $[1,[1,1]]=0$ are called 2-nilpotent.
Remark: All 2-nilpotent algebras are nilpotent by definition.

2-supernilpotent

Mal'cev algebras that satisfy $[1,1,1]=0$ are called 2-supernilpotent.

Example

$(\mathbb{Z}_4,+,2xyz)$ is 2-nilpotent, but not 2-supernilpotent.

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Some Properties of Ternary Commutators

Proposition (E. Aichinger, N. M., 2010)

(HC3) $[1,1,1] \leq [1,1]$
(HC8) $[1,[1,1]] \leq [1,1,1]$

Remark

In groups: $[1,[1,1]] = [1,1,1]$

Corollary of (HC8)

Every 2-supernilpotent Mal'cev algebra is 2-nilpotent.

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When is the commutator 0?

1. What does $[1, 1] = 0$ mean in groups?

Answer: Holds iff the group is commutative.

2. What does $[1, 1] = 0$ mean in Mal'cev algebras?

Theorem (Gumm, Hagemann, Herrmann)

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3. What does $[1, 1, 1] = 0$ mean in Mal'cev algebras?

Theorem

For a Mal'cev algebra **A** the following are equivalent:

1. **A** is 2-supernilpotent ($[1, 1, 1] = 0$)
2. **A** is polynomially equivalent to an expanded group $\mathbf{V} = (A, +, -, 0, F)$ such that
   i) F is a set of at most binary absorbing operations on **V**,
   ii) every absorbing operation in Pol(**V**) is distributive with respect to + on both arguments, and
   iii) **V** is 2-nilpotent.

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Some Useful Statements

Theorem (R. Freese, R.N. McKenzie)

Let **A** be a nilpotent Mal'cev algebra with a Mal'cev term d. Then, the function $x \mapsto d(x, a, b)$ is bijective for all $a, b \in A$.

Corollary

Let **A** be a nilpotent Mal'cev algebra with a Mal'cev term d and let $o \in A$. Then, for all $a_1, a_2, b_1, b_2 \in A$ there exist $x, y \in A$ such that $d(x, o, a_1) = b_1$ and $d(a_2, o, y) = b_2$.

Theorem (M. Suzuki)

Every semigroup $(G, +)$ such that the equations $a_1 + x = b_1$ and $y + a_2 = b_2$ are solvable, for all $a_1, a_2, b_1, b_2 \in A$, is a group.

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The Group Operation

Let us suppose that **A** is 2-supernilpotent. We show briefly that **A** has a polynomial group operation.

Let $o \in A$ and let d be a Mal'cev term of a Mal'cev algebra **A**. We define $+ : A^2 \to A$ by

$$x + y := d(x, o, y),$$

for all $x, y \in A$.

The idea

To prove that $+$ is a group operation we have to show that $+$ is associative.

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Proposition

p is an absorbing polynomial at (o,o,o) with value o.

Corollary

$(p(a,b,c),o) = (p(a,b,c),p(o,o,o)) \in [1,1,1] = 0$ for all $a,b,c \in A$.

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Associativity

We take $a, b, c \in A$.

$p(a, b, c) = o$ or equivalently

$$d(d(d(a, o, b), o, c), d(a, o, d(b, o, c)), o) = o.$$

$d(d(a, o, b), o, c) = d(a, o, d(b, o, c))$, because

$$x \mapsto d(x, d(a, o, d(b, o, c)), o).$$

is bijective. (A is 2-nilpotent by (HC8))

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Thank You for the Attention!