Duality via truth for distributive interlaced bilattices

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The 4th Novi Sad Algebraic Conference & Semigroups and Applications 2013

Motivations

- Duality via truth (DvT): duality between classes of algebras and classes of relational systems (so called frames).
- We view algebras and frames as semantical structures for formal languages.
- A duality principle: a given class of algebras and a class of frames provide equivalent semantics in the sense that a formula α (resp. a sequent $\alpha \vdash \beta$ - a pair of formulas where under the assumption of α the conclusion of β is provable) is true with respect to one semantics iff it is true with respect to the other semantics.
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We consider algebras with distributive lattice redact.

Priestley duality for distributive lattices

Priestley proved that the category of bounded distributive lattices and the category of compact totally order disconnected spaces $(X, \leq, \tau)$ (Priestley spaces) are dually equivalent.

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Let $\mathcal{A}lg$ be a class of algebras and let $\mathcal{F}rm$ be a class of frames.

Step 1. With every frame $X \in \mathcal{F}rm$ associate its *complex algebra* $\mathfrak{C}m(X)$ of X and show that $\mathfrak{C}m(X) \in \mathcal{A}lg$.
Step 2. With every algebra $L \in \mathcal{A}lg$ associate its *canonical frame* $\mathfrak{C}f(L)$ and show that $\mathfrak{C}f(L) \in \mathcal{F}rm$.
Step 3. Prove

Representation theorem for algebras and frames

1. Every algebra $L \in \mathcal{A}lg$ is embeddable into the complex algebra of its canonical frame, $\mathfrak{C}m(\mathfrak{C}f(L))$.
2. Every frame $X \in \mathcal{F}rm$ is embeddable into the canonical frame of its complex algebra, $\mathfrak{C}f(\mathfrak{C}m(X))$.

The general method [Orłowska, Radzikowska]

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The general method (cont.)

Step 4. Duality via truth

1. Define a propositional language $\mathcal{L}an_{Alg}$ over the set *Var* of propositional variables.
2. A sequent $\alpha \vdash \beta$ is true in an algebra L whenever $v(\alpha) \leq v(\beta)$ for any assignment $v : Var \to L$ extended for all the formulas of $\mathcal{L}an_{Alg}$;
   it is $\mathcal{A}lg$-valid whenever it is true in every $L \in \mathcal{A}lg$.
3. For any $X \in \mathcal{F}rm$, define $\mathcal{M} = (X, m)$ where $m : Var \to 2^X$. Extend m to all formulas in such a way that m is a valuation in the complex algebra $\mathfrak{C}m(X)$ of X.
4. A sequent $\alpha \vdash \beta$ is true in $\mathcal{M}$ if $m(\alpha) \subseteq m(\beta)$;
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The general method (cont.)

Step 5.
Establish DvT between the classes $\mathcal{A}lg$ and $\mathcal{F}rm$.

Duality via truth

For every sequent $\alpha \vdash \beta$ of $\mathcal{L}an_{\mathcal{A}lg}$ the following statements are equivalent:
(a) $\alpha \vdash \beta$ is $\mathcal{A}lg$-valid;
(b) $\alpha \vdash \beta$ is $\mathcal{F}rm$-valid.

Pre-bilattices

A pre-bilattice is an algebra $L = (L, \wedge, \vee, \sqcap, \sqcup)$ where $L = (L, \wedge, \vee)$ and $L = (L, \sqcap, \sqcup)$ are lattices with respective orders $\leq_t$ and $\leq_k$.

A pre-bilattice is:

- interlaced whenever each one of the four operations $\{\wedge, \vee, \sqcap, \sqcup\}$ is monotonic with respect to both orders $\leq_t$ and $\leq_k$.
- distributive whenever each one of twelve lattice redacts is distributive.
- bounded whenever each one of two lattice $(L, \leq_t)$ and $(L, \leq_k)$is bounded.

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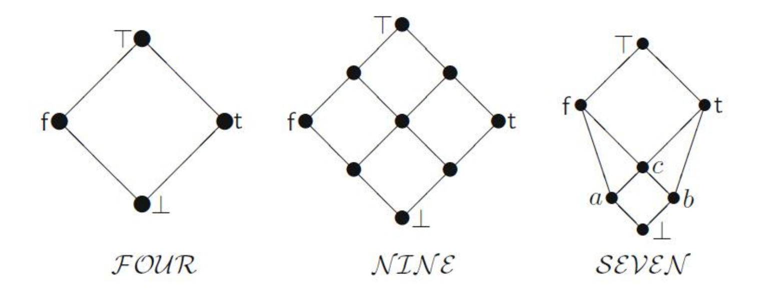
Examples of bilattices



pB-lattices

Any bounded distributive interlaced pre-bilattice $(L, \wedge, \vee, \sqcap, \sqcup, 0, 1, \bot, \top)$ may be viewed as a bounded distributive lattice [Avron] endowed with two complementary constants, that is a structure of the form $(L, \wedge, \vee, 0, 1, \bot, \top)$ where

$$\top \wedge \bot = 0$$
$$\top \vee \bot = 1.$$

This structure will be referred to as *pB-lattice*.

pB-frames

A *pB-frame* is a system $(X, \leq, \Delta)$ where $(X, \leq)$ is a poset, $\Delta \subseteq X$, and for all $x, y \in X$,

$$x \leq y \;\Rightarrow\; (x \in \Delta \Leftrightarrow y \in \Delta).$$

The *complex algebra of a pB-frame* $(X, \leq, \Delta)$ is a system $(L_X, \cap, \cup, \emptyset, X, \bot_\Delta, \top_\Delta)$ such that

$$L_X := \{A \subseteq X : A = \uparrow A\}$$
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Proposition

The complex algebra of a pB-frame is a pB-lattice.

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Canonical frames of pB-lattices

The *canonical frame of a pB-lattice* $(L, \wedge, \vee, 0, 1, \bot, \top)$ is a relational system $(X_L, \subseteq, \Delta_L)$ such that X_L is a set of all prime filters of $(L, \wedge, \vee, 0, 1)$ and

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Representations for pB-lattices and pB-frames

Let $h : L \to L_{X_L}$ be defined as $h(a) := \{F \in X_L : a \in F\}$ and let $k : X \to X_{L_X}$ be defined as $k(x) := \{A \subseteq X : x \in A\}$.

Theorem

(a) Every pB-lattice is embeddable into the complex algebra of its canonical frame.

(b) Every pB-frame is embeddable into the canonical frame of its complex algebra.

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DvT for pB-lattices

Let Lan_{pB} be a propositional language built up from a countable set of propositional variables Var using conjunction $\wedge$ and disjunction $\vee$ and four constants t, f, T and F.

Let Alg_{pB} be the class of pB-lattices and let $L \in Alg_{pB}$. A *valuation in* L is a mapping $v : Var \to L$ such that $v(t) = 1$, $v(T) = \top$, $v(f) = 0$ and $v(F) = \bot$ extended to the set of all formulas as usual:

$$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$$
$$v(\alpha \vee \beta) = v(\alpha) \vee v(\beta).$$

A sequent $\alpha \vdash \beta$ is Alg_{pB}-valid iff for every $L \in Alg_{pB}$ and for every valuation v in L, $v(\alpha) \leq v(\beta)$.

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DvT for pB-lattices (cont.)

Let $X = (X, \leqslant, \Delta)$ be a pB-frame. A model based on X is a system $M = (X, m)$ where $m : Var \to L_X$ is such that $m(t) = X$, $m(f) = \emptyset$, $m(T) = \Delta$ and $m(F) = -\Delta$.

The satisfaction relation $\models$ is defined for all formulas of $\mathcal{L}an_{pB}$

$$M, x \models p \;\Leftrightarrow\; x \in m(p) \text{ for every } p \in Var$$
$$M, x \models \alpha \wedge \beta \;\Leftrightarrow\; M, x \models \alpha \text{ and } M, x \models \beta$$
$$M, x \models \alpha \vee \beta \;\Leftrightarrow\; M, x \models \alpha \text{ or } M, x \models \beta.$$

Put $m(\alpha) = \{x \in X : M, x \models \alpha\}$.

Note: m is a valuation in the complex algebra $\mathfrak{C}m(X)$ of X.

$\alpha \vdash \beta$ is $\mathcal{F}rm_{pB}$-valid iff for every $X \in \mathcal{F}rm_{pB}$ and for every m, $m(\alpha) \subseteq m(\beta)$.

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(a) A sequent $\alpha \vdash \beta$ is $\mathcal{A}lg_{pB}$–valid;

(b) A sequent $\alpha \vdash \beta$ is $\mathcal{F}rm_{pB}$–valid.

Further works

- We also developed DvT for
  - bilattices (pre-bilattices + true order reversing and knowledge order preserving involution)
  - bilattices with conflation (bilattices + knowledge order reversing and true order preserving involution).
- In work: DvT for bilattices with Heyting implication and residuated bilattices.
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Further works

- We also developed DvT for
  - bilattices (pre-bilattices + true order reversing and knowledge order preserving involution)
  - bilattices with conflation (bilattices + knowledge order reversing and true order preserving involution).
- In work: DvT for bilattices with Heyting implication and residuated bilattices.
- Future work: DvT for various classes of bilattices of significience in CS.