Asymmetric regular types

Slavko Moconja
Joint work with Predrag Tanović

Faculty of Mathematics, Belgrade, Serbia

Invariant types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global type, and small $A \subset \overline{\mathrm{M}}$.

Type $p(x)$ is $A-$invariant if $f(p) = p$, for every $f \in \mathrm{Aut}_A(\overline{\mathrm{M}})$.

Invariant types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global type, and small $A \subset \overline{\mathrm{M}}$.

Type $p(x)$ is $A-$invariant if $f(p) = p$, for every $f \in \mathrm{Aut}_A(\overline{\mathrm{M}})$.

Fact. If $p(x)$ is $A-$invariant and $B \supseteq A$, then $p(x)$ is $B-$invariant.

Regular types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global non-algebraic type and small $A \subset \overline{\mathrm{M}}$.

Pair $(p(x), A)$ is regular if:

1. $p(x)$ is $A-$invariant and
2. for every $a \models p \mid A$ and every small $B \supseteq A$: either $a \models p \mid B$ or $p \mid B \vdash p \mid Ba$.

Regular types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global non-algebraic type and small $A \subset \overline{\mathrm{M}}$.

Pair $(p(x), A)$ is regular if:

1. $p(x)$ is $A-$invariant and
2. for every $a \models p \mid A$ and every small $B \supseteq A$: either $a \models p \mid B$ or $p \mid B \vdash p \mid Ba$.

Fact. If $(p(x), A)$ is a regular pair and $B \supseteq A$, then $(p(x), B)$ is a regular pair.

Asymmetric types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global non-algebraic $A-$invariant type.

Type $p(x)$ is asymmetric if for some $B \supseteq A$ and Morley sequence (a, b) in p over B: $ab \not\equiv ba\,(B)$.

Asymmetric types

Let $p(x) \in \mathrm{S}_1(\overline{\mathrm{M}})$ be a global non-algebraic $A-$invariant type.

Type $p(x)$ is asymmetric if for some $B \supseteq A$ and Morley sequence (a, b) in p over B: $ab \not\equiv ba\,(B)$.

Theorem

Suppose that pair $(p(x), A)$ is regular and $p(x)$ is asymmetric. Then there exists a finite extension A_0 of A and A_0-definable partial order $\leq$ such that every Morley sequence in p over A_0 is strictly increasing.

A. Pillay, P. Tanović, *Generic stability, regularity and quasiminimality*

$\mathrm{cl}_{p,A}$

Let $(p(x), A)$ be a regular pair. Assume that $p(x)$ is asymmetric over A.

For $X \subseteq (p|A)(\overline{\mathrm{M}})$ we define closure $\mathrm{cl}_{p,A}(X) \subseteq (p|A)(\overline{\mathrm{M}})$ with:

$$\mathrm{cl}_{p,A}(X) = \{a \models p|A \mid a \not\models p|AX\}.$$

For small $B \subset (p|A)(\overline{\mathrm{M}})$ we set:

$$\mathrm{cl}_{p,A,B}(X) = \mathrm{cl}_{p,A}(BX).$$

Also, if M is some small model that contains A we define:

$$\mathrm{cl}^{\mathrm{M}}_{p,A}(X) = \mathrm{cl}_{p,A}(X) \cap \mathrm{M} \text{ and } \mathrm{cl}^{\mathrm{M}}_{p,A,B}(X) = \mathrm{cl}_{p,A,B}(X) \cap \mathrm{M}.$$

$\mathrm{scl}_{p,A}$

For $a \models p|A$ we define symmetric closure $\mathrm{scl}_{p,A}(a) \subseteq (p|A)(\overline{\mathrm{M}})$ with:

$$\mathrm{scl}_{p,A}(a) = \{b \in \mathrm{cl}_{p,A}(a) \mid a \in \mathrm{cl}_{p,A}(b)\}.$$

For $X \subseteq (p|A)(\overline{\mathrm{M}})$ we define symmetric closure $\mathrm{scl}_{p,A}(X) \subseteq (p|A)(\overline{\mathrm{M}})$ with:

$$\mathrm{scl}_{p,A}(X) = \bigcup_{a \in X} \mathrm{scl}_{p,A}(a).$$

We also define $\mathrm{scl}_{p,A,B}$, $\mathrm{scl}^{\mathrm{M}}_{p,A}$ and $\mathrm{scl}^{\mathrm{M}}_{p,A,B}$.

Some facts about $\mathrm{cl}_{p,A}$ and $\mathrm{scl}_{p,A}$

1. $p|AX \vdash p|A\mathrm{cl}_{p,A}(X)$;
2. $\mathrm{cl}_{p,A}$ $(\mathrm{cl}_{p,A,B})$ is closure operator on $(p|A)(\overline{\mathrm{M}})$;
3. $\mathrm{cl}_{p,A}(a_1, a_2, \dots, a_n) = \mathrm{cl}_{p,A}(a)$, where a is any maximal element in $\{a_1, a_2, \dots, a_n\}$;
4. (a, b) is Morley sequence in p over AB iff $a \notin \mathrm{cl}_{p,A}(B)$ and $b \notin \mathrm{cl}_{p,A}(Ba)$;
5. $\mathrm{cl}_{p,A}(X) = \bigcup_{(\exists x \in X) a \leq x} \mathrm{scl}_{p,A}(a)$;
6. $(p|A)(\overline{\mathrm{M}})/\mathrm{scl}_{p,A} = \{\mathrm{scl}_{p,A}(a) \mid a \vDash p|A\}$ is a partition of $(p|A)(\overline{\mathrm{M}})$;
7. $(p|A)(\mathrm{M})/\mathrm{scl}^{\mathrm{M}}_{p,A} = \{\mathrm{scl}^{M}_{p,A}(a) \mid a \vDash p|A\}$ is a partition of $(p|A)(\mathrm{M})$ (M is small model that contains A).

Order on $(p|A)(\overline{\mathrm{M}})/\mathrm{scl}_{p,A}$

Lemma

Suppose that $\mathrm{scl}_{p,A}(a) \neq \mathrm{scl}_{p,A}(b)$ *and* $a < b$. *Then for every* $x \in \mathrm{scl}_{p,A}(a)$ *and* $y \in \mathrm{scl}_{p,A}(b)$ *is* $x < y$.
If $\mathrm{scl}_{p,A}(a) \neq \mathrm{scl}_{p,A}(b)$ *and* $a \not< b$, *then* $b < a$.

Corollary. Set $(p|A)(\overline{\mathrm{M}})/\mathrm{scl}_{p,A}$ is linearly ordered.

Order on $(p|A)(\overline{\mathrm{M}})/\mathrm{scl}_{p,A}$

Lemma

Suppose that $\mathrm{scl}_{p,A}(a) \neq \mathrm{scl}_{p,A}(b)$ *and* $a < b$. *Then for every* $x \in \mathrm{scl}_{p,A}(a)$ *and* $y \in \mathrm{scl}_{p,A}(b)$ *is* $x < y$.
If $\mathrm{scl}_{p,A}(a) \neq \mathrm{scl}_{p,A}(b)$ *and* $a \not< b$, *then* $b < a$.

Corollary. Set $(p|A)(\overline{\mathrm{M}})/\mathrm{scl}_{p,A}$ is linearly ordered.

Lemma

Maximal Morley sequence in p *over* A *in some small model* M *that contains* A *is exactly any set of representatives of* $(p|A)(\mathrm{M})/\mathrm{scl}^{\mathrm{M}}_{p,A}$ *partition.*

Corollary. Any two maximal Morley sequences in p over A in M have the same order-type.

Non-definable $\mathrm{scl}_{p,A}$

Theorem

Assume that $\mathrm{scl}_{p,A}(a)$ is not $Aa-$definable, for some (every) $a \in (p|A)(\overline{\mathrm{M}})$. Then, for every countably order type there exists a countable model M such that the maximal Morley sequence in p over A in M has that order type.

Corollary. If there exists global $A-$invariant, regular and asymmetric type whose $\mathrm{scl}_{p,A}$ is not $Aa-$definable, then there are $2^{\aleph_0}$ non-isomorphic countable models.

Example of asymmetric regular types

Let M be a model of small $o-$minimal theory, $p \in \mathrm{S}_1(A)$ non-algebraic type, and $\overline{\mathrm{M}}$ monster model.

Fact. $p(\overline{\mathrm{M}})$ is convex set.

We have four kinds of p:

- (isolated type) there exist $c, d \in \mathrm{dcl}(A)$ such that $c < x < d \vdash p(x)$;
- (non-cut) there exist $c \in \mathrm{dcl}(A)$ and strictly decreasing sequence (d_n) in $\mathrm{dcl}(A)$ such that $\{c < x < d_n \mid n \in \omega\} \vdash p(x)$;
- (non-cut) there exist strictly increasing sequence (c_n) in $\mathrm{dcl}(A)$ and $d \in \mathrm{dcl}(A)$ such that $\{c_n < x < d \mid n \in \omega\} \vdash p(x)$;
- (cut) there exist strictly increasing sequence (c_n) and strictly decreasing sequence (d_n) in $\mathrm{dcl}(A)$ such that $\{c_n < x < d_n \mid n \in \omega\} \vdash p(x)$.

Left and right global extensions: Case I

Assume that there exists $c \in \mathrm{dcl}(A)$ such that c determines p "on the left side". Then for every $\overline{\mathrm{M}}-$formula ϕ, either ϕ or $\neg\phi$ has interval that contains (c, t), for some $t \in p(\overline{\mathrm{M}})$.

We define left global extension of p:
$p_L(x) = \{\phi(x) \mid \phi(\overline{\mathrm{M}}) \text{ contains } (c, t), \text{ for some } t \in p(\overline{\mathrm{M}})\} \in \mathrm{S}_1(\overline{\mathrm{M}})$.

Similarly we define right global extension p_R of p, if there exists $d \in \mathrm{dcl}(A)$ such that d determines p "on the right side".

Left and right global extensions: Case II

Assume that there exists strictly increasing sequence (c_n) such that (c_n) determines p "on the left side". Then for every $\overline{\mathrm{M}}-$formula ϕ, either ϕ or $\neg\phi$ has interval that contains all but finitely many c_n.

We define left global extension of p:
$p_L(x) = \{\phi(x) \mid \phi(\overline{\mathrm{M}}) \text{ contains all but finitely many } c_n\} \in \mathrm{S}_1(\overline{\mathrm{M}})$.

Similarly we define right global extension p_R of p, if there exists strictly decreasing sequence (d_n) such that (d_n) determines p "on the right side".

p_L and p_R

Theorem

Both p_L and p_R are $A-$invariant, regular and asymmetric extensions of p.

Moreover, p_L and p_R are the only two global $A-$invariant extensions of p.

Any Morley sequence in p_R is strictly increasing, and any Morley sequence in p_L is strictly decreasing.

$\mathrm{scl}_{p_L,A}$, $\mathrm{scl}_{p_R,A}$

Lemma

Let $a \in p(\overline{\mathrm{M}})$. *Then:*
$\mathrm{scl}_{p_L,A}(a) = \mathrm{scl}_{p_R,A}(a) =$ *convex closure* $(\mathrm{dcl}(Aa) \cap p(\overline{\mathrm{M}}))$.

Corollary. $I \subset p(\overline{\mathrm{M}})$ is a Morley sequence in p_L over A in $\overline{\mathrm{M}}$ iff it is Morley sequence in p_R over A in $\overline{\mathrm{M}}$. Also, $I \subseteq p(M)$ is a maximal Morley sequence in p_L over A in M iff it is maximal Morley sequence in p_R over A in M, for any small model M that contains A.

Remark. If $p \in \mathrm{S}_1(A)$, then for some (any) $a \in p(\overline{\mathrm{M}})$, $\mathrm{scl}_{p_L,A}(a)$ is $Aa-$definable iff $\mathrm{scl}_{p_L,A}(a) = \{a\}$.

$\perp^w$, $\not\perp^w$, dimension

Let p, q be two complete types (with parameters). We say that $p \perp^w q$ iff $p(\overline{x}) \cup q(\overline{y}) \vdash \mathrm{tp}(\overline{xy})$.

$\not\perp^w$ is equivalence relation on $\mathrm{S}_1(\emptyset)$. Let $\{p_i \mid i \in I\}$ be the set of non-algebraic representatives of this equivalence relation.

$\perp^w$, $\not\perp^w$, dimension

Let p, q be two complete types (with parameters). We say that $p \perp^w q$ iff $p(\overline{x}) \cup q(\overline{y}) \vdash \mathrm{tp}(\overline{xy})$.

$\not\perp^w$ is equivalence relation on $\mathrm{S}_1(\emptyset)$. Let $\{p_i \mid i \in I\}$ be the set of non-algebraic representatives of this equivalence relation.

Let M be any countable model, $A_i =$ maximal Morley sequence in p_{iL}, and $A = \bigcup_{i \in I} A_i$.

Theorem

M *is prime over* A.

M *and* N *are isomorphic iff maximal Morley sequence in* p_{iL} *in* M, *and maximal Morley sequence in* p_{iL} *in* N *have the same order-type, for every* $i \in I$.

Additional assumption

Assume that there are $< 2^{\aleph_0}$ countable models.

Then $\mathrm{scl}_{p_{iL},\emptyset}(a) = \{a\}$, for every type p_i.

Let M be a countable model. Under this assumption if p is:

1. algebraic type, then $p(\mathrm{M})$ is a point;
2. isolated type, then $p(\mathrm{M})$ is $\mathbb{Q}$;
3. non-cut, then there are 3 possibilities for $p(\mathrm{M})$;
4. cut, then there are 6 possibilities for $p(\mathrm{M})$.

Since there are $< 2^{\aleph_0}$ countable models, there are only finitely many non-isolated types in $\{p_i \mid \in I\}$. If m of them are cuts, and n of them are non-cuts, then there are exactly $6^m 3^n$ countable models.

Additional assumption

Assume that there are $< 2^{\aleph_0}$ countable models.

Then $\mathrm{scl}_{p_{iL},\emptyset}(a) = \{a\}$, for every type p_i.

Let M be a countable model. Under this assumption if p is:

1. algebraic type, then $p(\mathrm{M})$ is a point;
2. isolated type, then $p(\mathrm{M})$ is $\mathbb{Q}$;
3. non-cut, then there are 3 possibilities for $p(\mathrm{M})$;
4. cut, then there are 6 possibilities for $p(\mathrm{M})$.

Since there are $< 2^{\aleph_0}$ countable models, there are only finitely many non-isolated types in $\{p_i \mid \in I\}$. If m of them are cuts, and n of them are non-cuts, then there are exactly $6^m 3^n$ countable models.

Laura Mayer, *Vaught's Conjecture for o−Minimal Theories*

Thank you for your attention!