Algebraic Models of Computation

Monoids Are Omnipotent

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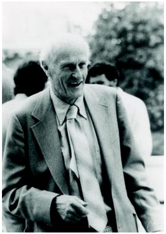
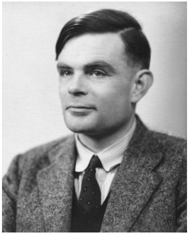
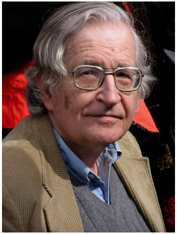
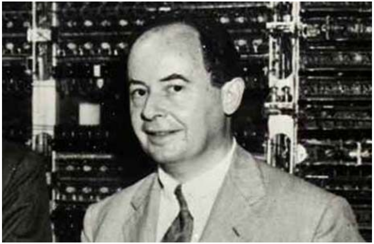
History

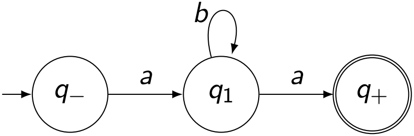




Automata

An automaton consists of:

- A digraph of *states* and *transition*;
- Edge labels in a finite alphabet Σ;
- A distinguished start vertex (arrow in);
- A set of distinguished end vertices (drawn with double outlines).

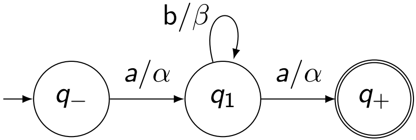


M-automata

An M-automaton $\mathcal{A}$ consists of:

- An *underlying* automaton $\mathcal{B}$;
- A register with values in M (initialises to 1);
- A *transition valence* in M for each transition arrow in $\mathcal{B}$.

Transition: right-multiply transition valence onto register. Accept on $\mathcal{B}$-accept state *only* with 1 on register.



$$M = \langle\ \alpha,\ \beta\ \mid\ \alpha^2 = \beta^3 = 1\ \rangle$$

Rational Transductions

The following relations in $\Sigma^* \times \Xi^*$ are rational transductions:

- The finite relations;
- Finite unions of rational transductions;
- Products in $\Sigma^* \times \Xi^*$ of rational transductions;
- Submonoids in $\Sigma^* \times \Xi^*$ generated by rational transductions.

Nivat's Theorem *Let $\tau : \Sigma^* \rightharpoonup \Xi^*$ (be a rational transduction). Then there exists an alphabet Θ and letter-to-letter morphisms $f : \Theta^* \to \Sigma^*$, $g : \Theta^* \to \Xi^*$ and a local regular language $K \subseteq \Theta^*$ such that*

$$\tau(x) = g(f^{-1}(x) \cap K)$$

M-Languages

Proposition *(Kambites–Render '06) The following are equivalent for $L \subseteq \Sigma^*$, if M is finitely generated:*

- *L is accepted by an M-automaton;*
- *L is a rational transduction of M's identity language w.r.t. some finite $A \subset M$;*
- *L is a rational transduction of M's identity language w.r.t. any finite $A \subset M$.*

Corollary *Let N and M be finitely generated monoids. Then the identity language of N is accepted by an M automaton precisely if every language accepted by an N-automaton is accepted by an M-automaton.*

Equivalence of M and N

Proposition *(KR '06) Let L be decided by an M-automaton. Then L is recognised by an N-automaton for some finitely generated $N \leq M$.*

Proposition *(KR '07) Let $I \trianglelefteq M$ be an ideal. If L is recognised by an M-automaton then L is recognised by M/I.*

Corollary *"It suffices to consider 0-simple monoids when computing with monoids."*

Determinism

An M-automaton is (strongly) deterministic if no letter labels two transitions from a given state, and determinisable if it decides the same language as some deterministic M-automaton.

Proposition*(Zetzsche) The following are equivalent for a fixed monoid M:*

- *Each finitely generated submonoid $N \leq M$ has finitely many elements in $[1]_{\mathcal{J}_N}$ (equiv. $[1]_{\mathcal{R}_N}$, $[1]_{\mathcal{L}_N}$);*
- *M-automata are determinisable;*
- *M-automata recognise precisely the regular languages.*

Decision Power

Theorem *Let L be a language. There exists a monoid $M := M_L$ and an M-automaton that recognises it.*

Corollary *Let $\mathcal{L}$ be a class of languages. There exists a monoid $M_{\mathcal{L}}$ recognising each language in $\mathcal{L}$.*
If $\mathcal{L}$ is a rational cone then $\mathcal{L}$ is precisely the class of languages recognised by $M_{\mathcal{L}}$.