On multisemigroups

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Definition

- S — a set
- $m : S \times S \to \mathcal{P}(S)$ — a map
- m is called 'multivalued multiplication'
- (S, m) is a multisemigroup if m is associative: for any $a, b, c \in S$

$$\bigcup_{x \in m(a,b)} m(x, c) = \bigcup_{x \in m(b,c)} m(a, x)$$

- We write $a \cdot b$ or $a \circ b$ or ab etc. instead of $m(a, b)$
- Any semigroup is a multisemigroup whose multiplication is single-valued
- A multisemigroup $(S, *)$ is called a hypergroup if the reproduction axiom holds: $S * a = a * S = S$ for all $a \in S$.

Some history

- Definition of multistructures — F. Marty, 1934 (at least).
- H. Campaigne, M. Dresher, O. Ore, H. S. Wall in 1930th, more recent M. Koskas, A. Hasankhani, T. Vougiouklis, M. Krasner, M. Marshall, O. Viro and many others....
- Multirings, multifields: M. Krasner, 1956, M. Marshall 2006, O. Viro, 2010.
- V Mazorchuk and V. Miemietz: multisemigroups appear naturally in higher representation theory and categorification, 2011, 2012.

First examples

One-element multisemigroups

$S = \{a\}$. (i) $a * a = a$, (ii) $a * a = \varnothing$.

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Inflation of multisemigroups

$(S, *)$ — a multisemigroup, X — a set, $f : X \to S$ — a surjection. For $x, y \in X$ define $x *_f y := \{z \in X \mid f(z) \in f(x) * f(y)\}$.
$(X, *_f)$ is a multisemigroup called *inflation* of S with respect to f.

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The trivial multisemigroups

For any S there are two trivial multisemigroup structures on S: $(S, \diamond)$ and $(S, \bullet)$: $s \diamond t := \emptyset$ for all $s, t \in S$ — inflation of (ii), $s \bullet t := S$ for all $s, t \in S$ — inflation of (i).

Reproductive construction

- $(S, \cdot)$ — a semigroup.
- $f : S \to \mathcal{P}(S)$ — a map.
- For $a, b \in S$ define $a * b := f(a)f(b)$.
- If for any $a, b \in S$ we have $f(f(a)f(b)) = f(a)f(b)$, then $(S, *)$ is a multisemigroup.
- Indeed, $(a * b) * c = a * (b * c) = f(a)f(b)f(c)$.

Reproductive construction: examples

- $(G, \cdot)$ — a group, $H < G$. $f : G \to \mathcal{P}(G)$, given by $a \mapsto Ha$, satisfies the reproductive condition, so $(G, *)$ is a multisemigroup.

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- $(M, \cdot)$ — a monoid. $f : S \to \mathcal{P}(M)$, given by $a \mapsto Ma$, satisfies the reproductive condition, so $(M, *)$ is a multisemigroup.

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- A — an alphabet. $f : A^* \to \mathcal{P}(A^*)$, sending u to the set of all its scattered subwords, satisfies the reproductive condition, so $(A^*, *)$ is a multisemigroup.

Construction from interassociates and variants

- $(S,\cdot)$ — a (multi)semigroup. A (multi)semigroup $(S,\circ)$ is called an interassociate of $(S,\cdot)$ if for any $a,b,c \in S$:

$$(a\cdot b)\circ c = a\cdot(b\circ c) \quad \text{and} \quad (a\circ b)\cdot c = a\circ(b\cdot c).$$

- For $a,b \in S$ set $a * b := (a\cdot b)\cup(a\circ b)$. $(S,*)$ is a multisemigroup.
- For example: $(S,\bowtie)$ — a (multi)semigroup, $X, Y \subseteq S$. For $a,b \in S$ set

$$a\cdot b := a\bowtie X\bowtie b, \quad \text{and} \quad a\circ b := a\bowtie Y\bowtie b.$$

  $(S,\cdot)$ and $(S,\circ)$ are variants of $(S,\bowtie)$ and each of them is an interassociate of the other.

- A — an associative algebra over some subring of real numbers.
- Assume that A has a basis $\mathbf{a} := \{a_i \mid i \in S\}$ with non-negative structure constants:

$$a_i a_j = \sum_{k \in S} c_{i,j}^k a_k \quad \text{and} \quad c_{i,j}^k \geq 0 \quad \text{for all} \quad i, j, k \in S.$$

- Define $*$: for $i, j \in S$ set

$$i * j := \{k \mid c_{i,j}^k > 0\}.$$

- $(S, *)$ is a multisemigroup.
- A similar construction works for the Boolean semiring $\mathbb{B} := \{0, 1\}$.

Connection with quantales

- $(S, *)$ — multisemigroups. $\mathcal{P}(S)$ inherits the natural structure of a semigroup by setting, for $A, B \in \mathcal{P}(S)$,

$$A * B := \bigcup_{a \in A,\, b \in B} a * b.$$

- $(\mathcal{P}(S), *)$ — semigroup. Moreover,

$$A * (\cup_i B_i) = \cup_i (A * B_i) \quad \text{and} \quad (\cup_i B_i) * A = \cup_i (B_i * A).$$

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- So $(\mathcal{P}(S), *)$ is a quantale (a sup-lattice with an associative product, which distributes over arbitrary joins).
- Conversely, if $(Q, \leq, *)$ is a quantale such that $(Q, \leq)$ is a complete atomic Boolean algebra, then it induces the structure of a multisemigroup on the set $S = S(Q)$ of atoms of Q.
- So multisemigroups can be viewed at as complete atomic Boolean algebras with quantale structure.

Homomorphisms

Let $(S, *)$ and $(T, \bullet)$ be multisemigroups.

- A strong homomorphism from S to T is a map $\varphi : S \to T$ such that for any $a, b \in S$
$$\bigcup_{s \in a*b} \varphi(s) = \varphi(a) \bullet \varphi(b).$$
- A weak homomorphism from S to T is a map $\varphi : S \to \mathcal{P}(T)$ such that for any $a, b \in S$ we have
$$\bigcup_{s \in a*b} \varphi(s) = \varphi(a) \bullet \varphi(b).$$

The category of multisemigroups with strong (weak) homomorphisms is equivalent to the category of complete atomic Boolean quantales with frame (sup-lattice) quantale homomorphisms.

Multisemigroups of ultrafilters

- Inspired by M. Gehrke, S. Grigorieff, J.-E. Pin "A topological approach to recognition".
- M — monoid, $L \subseteq M$, $s, t \in M$. The quotient $s^{-1}Lt^{-1}$ of L is

$$s^{-1}Lt^{-1} = \{x \in M \colon sxt \in M\}.$$

- B Boolean algebra of subsets of M that is closed under quotients. The syntactic congruence on M: $u \sim_B v$ iff for each $L \in B$ we have $u \in L$ if and only if $v \in L$.
- $M/\sim_B$ is the syntactic monoid of B.
- Assume M is the syntactic monoid of B. $\hat{M}$ is the dual space of B. Its elements correspond to ultrafilters of B. Elements of M correspond to principal ultrafilters.
- The multiplication of M extends to a multisemigroup multiplication $*$ on $\hat{M}$: If $p, q \in \hat{M}$ we set

$$p * q = \{f \in \hat{M} \colon f \supseteq \{XY : X \in p, Y \in q\}^{\uparrow}\}.$$

Multisemigroups of ultrafilters: examples

- (see GGP, example 3.1) M — discrete monoid. $B = \mathcal{P}(M)$. Its syntactic monoid is M, and $\hat{M} = \beta(M)$. The multisemigroup multiplication $*$ on $\beta(M)$ is

$$p * q = \{f \in \beta(M) \colon f \supseteq \{XY : X \in p, Y \in q\}^{\uparrow}\}.$$

- (see GGP, example 3.2) $M = (\mathbb{Z}, +)$. B — the Boolean algebra of finite and cofinite subsets of M. Its syntactic monoid is M, and $\hat{M} = \mathbb{Z} \cup \{\infty\}$, $+$ extends to $\hat{+}$:

$\hat{+}$	i	∞
j	$\{i+j\}$	$\{\infty\}$
∞	$\{\infty\}$	$\mathbb{Z} \cup \{\infty\}$

Zero elements

An element $z \in S$ is called a zero element if for every $a \in S$ $a * z = z * a = z$. It is unique, if exists, denote it by 0.

- Let $(S, *)$ be a multisemigroup with zero 0 and assume that $S \neq \{0\}$. Then for any $a, b \in S$, $a * b \neq \varnothing$. Let $T := S \setminus \{0\}$ and for $a, b \in T$ set $a \bullet b := a * b \setminus \{0\}$. Then $(T, \bullet)$ is a multisemigroup

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- Let $(S, *)$ be a multisemigroup without zero. Put $S^0 := S \cup \{0\}$ (we assume $0 \notin S$) and for $a, b \in S^0$ define

$$a \bullet b := \begin{cases} a * b; & a, b \in S,\ a * b \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

  Then $(S^0, \bullet)$ is a multisemigroup with zero 0.
- So, without loss, we can consider multisemigroups without a zero element.

Ideals and Green's relations

- A subset $I \subset S$ is called a left ideal if for any $a \in I$ and $s \in S$, $s * a \subset I$.
- For every $a \in S$ the set $S^1 * a$ is called the principal left ideal generated by a.
- The left pre-order $\leq_L$: $a \leq_L b$ if and only if $S^1 * b \subset S^1 * a$.
- The definitions above can be modified to right and two-sided cases .
- One can define Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{J}$.
- S is called simple if for any $a \in S$, $S^1 * a * S^1 = S$, that is S has a unique $\mathcal{J}$-class.

A multisemigroup from a rectangular band

$(1,1)$	$(1,2)$
$(2,1)$	$(2,2)$

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$$(2,1)\cdot(1,2) = ?$$

A multisemigroup from a rectangular band

$(1,1)$	$(1,2)$
$(2,1)$	$(2,2)$

$(2,1)\cdot(1,2)=(2,2)$

in a rectangular band

A multisemigroup from a rectangular band

$(1, 1)$	$(1, 2)$
$(2, 1)$	$(2, 2)$

$$(2, 1) * (1, 2) = \{(2^{\downarrow}, 2^{\downarrow})\}$$
$$\{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

in a multisemigroup

A multisemigroup from a rectangular band

$(1,1)$	$(1,2)$
$(2,1)$	$(2,2)$

$$(2,1) * (1,2) = \{(2^{\downarrow}, 2^{\downarrow})\}$$
$$\{(1,1),(1,2),(2,1),(2,2)\}$$

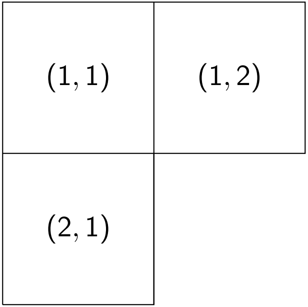
in a multisemigroup

$\{1,\dots,n\} \times \{1,\dots,n\}$: $(i,j)\cdot(k,l) = \{(p,q) \colon p \leq i, q \leq l\}$. What are the Green's relations? This finite multisemigroup is bisimple, but $S^1 * (1,1) \subsetneq S^1 * (1,2)$, and similarly for right principal ideals (this differs from what holds for semigroups!)

A multisemigroup with $\mathcal{L} \circ \mathcal{R} \neq \mathcal{R} \circ \mathcal{L}$

$(1,1)$	$(1,2)$
$(2,1)$	$(2,2)$

A multisemigroup with $\mathcal{L} \circ \mathcal{R} \neq \mathcal{R} \circ \mathcal{L}$



$$(i,j) \cdot (k,l) = (i,j) * (k,l) \setminus (2,2)$$

$$\{(2,1),(1,2)\} \in \mathcal{L} \circ \mathcal{R}$$

$$\{(2,1),(1,2)\} \notin \mathcal{R} \circ \mathcal{L}$$

Strongly simple multisemigroups

- We assume that S does not have 0 and that ideals are non-empty.
- An element $s \in S$ will be called a quark provided that $S^1 * s$ is a minimal left ideal and $s * S^1$ is a minimal right ideal.
- $Q(S)$ the set of all quarks in $(S, *)$ — the support of S.
- A simple multisemigroup $(S, *)$ will be called strongly simple if $S = Q(S)$.

Proposition

Let $(S, *)$ be a multisemigroup. If $(S, *)$ contains only one $\mathcal{H}$-class, then either $S \cong \mathbf{0}$ ($\mathbf{0} = \{0\}$ with $0 * 0 = \varnothing$) or S is a hypergroup.

Proposition

Assume $Q(S) \neq \varnothing$. Then:

(a) $Q(S)$ is a submultisemigroup.

(b) $Q(S)$ is the disjoint union of its intersections with $\mathcal{J}$-classes of S.

Structure of strongly simple multisemigroups

Theorem (Structure of strongly simple multisemigroups)

Let $(S, *)$ be a strongly simple multisemigroup.

(a) *For any $a, b \in S$: $\mathcal{L}_a \cap \mathcal{R}_b \neq \varnothing$.*

(b) *If H is an $\mathcal{H}$-class then either $H * H = \varnothing$ or H is a hypergroup.*

(c) *For $a, b \in S$: $a * b \neq \varnothing$ if and only if $\mathcal{L}_a \cap \mathcal{R}_b$ is a hypergroup.*

(d) *Assume $S \ncong \mathbf{0}$. Then every $\mathcal{L}$-class and every $\mathcal{R}$-class in S contains at least one hypergroup $\mathcal{H}$-class.*

(e) *Let $a\mathcal{R}b$ and $s \in S^1$ be such that $b \in a * s$. The map $x \mapsto x * s$ is a multivalued surjective map from $\mathcal{L}_a$ to $\mathcal{L}_b$ that preserves both $\mathcal{R}$- and $\mathcal{H}$-classes.*

(f) *Assume $S \ncong \mathbf{0}$. Let I be a minimal left ideal of S and J a minimal right ideal of S. Then $I \cap J = J * I$.*

A simple multisemigroup with not strongly simple support

$(1,1)$	$(1,2)$
$(2,1)$	$(2,2)$

$$(S,*):$$
$$(i,j)*(k,l)=\{(i^{\downarrow},l^{\downarrow})\}$$

A simple multisemigroup with not strongly simple support

$(1,1)$ $(1',1')$	$(1,2)$
$(2,1)$	$(2,2)$

$$T = S \cup (1',1')$$

$$\pi : T \to S :$$

$$\pi(a) = a, a \in S; \pi((1',1')) = (1,1)$$

$\circ : T \times T \to \mathcal{P}(T)$:

$$x \circ y = \begin{cases} \pi(x) * \pi(y), & x \in \{(1,1),(1',1'),(1,2)\} \text{ and} \\ & y \in \{(1,1),(1',1'),(2,1)\}; \\ (\pi(x) * \pi(y)) \cup (1',1'), & \text{otherwise .} \end{cases}$$

$(T, \circ)$ is a multisemigroup. $Q(T) = \{(1,1),(1',1')\}$. But $Q(T) \circ Q(T) = \{(1,1)\}$ and hence $Q(T)$ is not a hypergroup.