Finitely based finite algebras

Kearnes, Szendrei, Willard

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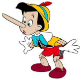
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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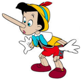
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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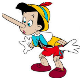
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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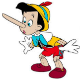
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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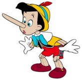
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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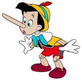

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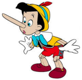
In fact, every finite group, ring, Lie algebra, or lattice is finitely based. Every algebra with only 2 elements is finitely based. 

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Words to know: algebra, identity, basis, variety.

Lyndon's groupoid, Murskii's groupoid

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Here is an groupoid that is not finitely based.

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	4	5	6	0	0	0
5	0	5	5	5	0	0	0
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Lyndon, 1954

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Here is another.

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

Murskii, 1965

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Thm. (Dolinka) The semiring of all binary relations on a set satisfying $1 < |X| < \infty$ is INFB.

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Words to know: subdirectly irreducible algebra, congruence, covering pair/atom/monolith.

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It is likely that Park was motivated by Baker's Theorem, which proves that the statement of Park's Conjecture is true for any variety whose members have distributive congruence lattices.

Baker's Theorem+extensions, in the language of TCT
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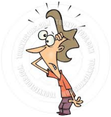

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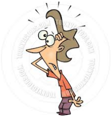

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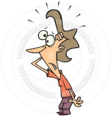

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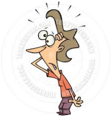

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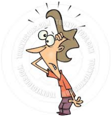

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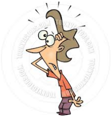

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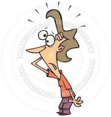

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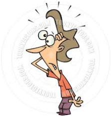
Part 2: the theorem



Thm. (Baker, 1977) Park's Conjecture is true for varieties whose types are in the set $\{\mathbf{3}, \mathbf{4}\}$.

Or, any congruence distributive variety with a finite residual bound is finitely axiomatizable.

Thm. (McKenzie, 1987) Park's Conjecture is true for varieties whose types are in the set $\{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$.

Or, any congruence modular variety with a finite residual bound is finitely axiomatizable.

Thm. (Willard, 2000) Park's Conjecture is true for varieties whose types are in the set $\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

Or, any variety omitting nontrivial abelian congruences, which has a finite residual bound, is finitely axiomatizable.

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There is a slightly more restrictive (but natural) condition: a term $d(x, y, z)$ is a *difference term* for $\mathcal{V}$ if $d(x, x, y) = y$ holds throughout $\mathcal{V}$ and $d(a, b, b) = a$ holds when $\mathrm{Cg}(a, b)$ is abelian.

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Thm. A variety that can be shown to be INFB using the shift automorphism method must contain an infinite subdirectly irreducible algebra.