Algebraic approach to coloring by oriented trees

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19% of definitions in this talk

- a relational structure: $\mathbb{A} = \langle A; R_1, \dots, R_n \rangle$, where $R_i \subseteq A^{k_i}$
- a di(rected)graph: $\mathbb{G} = \langle G; \rightarrow \rangle$, where $\rightarrow$ is binary
- an algebra: $\mathbf{A} = \langle A; \mathcal{F} \rangle$, $\mathcal{F}$ is a clone of operations on A
- a subuniverse ($C \leq \mathbf{A}$): a subset closed under all operations
- an idempotent algebra: every $f \in \mathcal{F}$ satisfies $f(x, x, \dots, x) \approx x$ (equivalently, $\{a\} \leq \mathbf{A}$ for every $a \in A$)

All domains in this talk are finite!

Fixed template CSPs

- fix a finite relational structure $\mathbb{A}$
- the Constraint satisfaction problem over $\mathbb{A}$ = membership problem for the set

$$\mathrm{CSP}(\mathbb{A}) = \{\mathbb{X} \mid \mathbb{X} \to \mathbb{A}\}$$

- goal: characterize relational structures wrt. complexity of the CSP and related algorithmic properties

Conjecture (The CSP dichotomy conjecture – Feder, Vardi '93)

For every $\mathbb{A}$, $\mathrm{CSP}(\mathbb{A})$ is in P or NP-complete.

Algebra of polymorphisms

- polymorphisms of $\mathbb{A}$ = operations preserving all relations

$$\begin{array}{ccccccc} f(a_1 & a_2 & \dots & a_n) & = & a \\ \downarrow & \downarrow & & \downarrow & \Longrightarrow & \downarrow \\ f(b_1 & b_2 & \dots & b_n) & = & b \end{array}$$

- $\langle A; \mathrm{Pol}(\mathbb{A})\rangle$ = the algebra of polymorphisms of $\mathbb{A}$
- a primitive positive (pp-) formula: $\exists$, $\wedge$, $=$
- relations pp-definable from $\mathbb{A}$ = $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$

Relational structures are algebras, too!
(See Ross Willard's talk.)

The algebraic approach to CSP & Maltsev conditions

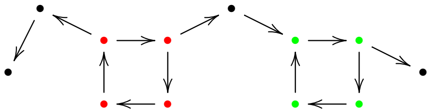
- Bulatov, Jeavons, Krokhin '00-'05: complexity of $\mathtt{CSP}(\mathbb{A})$ is controlled by the equational theory of $\mathrm{HSP}(\mathbb{A})$
- a strong Maltsev condition = finite set of equations in some operation symbols
- a weak near-unanimity (WNU) = n-ary operation ($n \geq 2$) satisfying

$$f(x, \dots, x, y) \approx f(x, \dots, x, y, x) \approx \dots \approx f(y, x, \dots, x)$$

- a near-unanimity (NU) = a WNU such that $f(x, \dots, x, y) \approx x$
- a majority = ternary NU (eg. $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$)
- a semilattice operation

Cores & constants

- a core structure = every endomorphism is an automorphism
- every structure has a unique (up to isomorphism) core



- the algebraic approach works only for cores
- but $\mathtt{CSP}(\mathbb{A}) = \mathtt{CSP}(\text{core of } \mathbb{A})$
- also, we can add all singleton unary relations (i.e., we can prescribe values to variables) $\Rightarrow$ idempotent algebras

Two important classes of algebras

- Taylor algebra = satisfies any nontrivial strong Maltsev condition
- Maróti, McKenzie '06: Taylor iff has some WNU
- Bulatov, Jeavons, Krokhin '00-05:
  If a core $\mathbb{A}$ is not Taylor, then $\mathtt{CSP}(\mathbb{A})$ is $\mathtt{NP}$-complete.
- **Algebraic dichotomy conjecture**
  If a core $\mathbb{A}$ is Taylor, then $\mathtt{CSP}(\mathbb{A})$ is in $\mathtt{P}$.
- $\mathbb{A}$ has bounded width (BW) = $\mathtt{CSP}(\mathbb{A})$ solvable by local consistency checking (in $\mathtt{P}$), "Can't encode linear equations."
- *SD*($\wedge$) algebra = $\mathrm{HSP}(\mathbf{A})$ has $\wedge$-semidistributive congruence lattices
- Maróti, McKenzie '06: *SD*($\wedge$) iff has WNUs of almost all arities
- Barto, Kozik '08: **Bounded width theorem**
  A core $\mathbb{A}$ has BW iff it is *SD*($\wedge$).

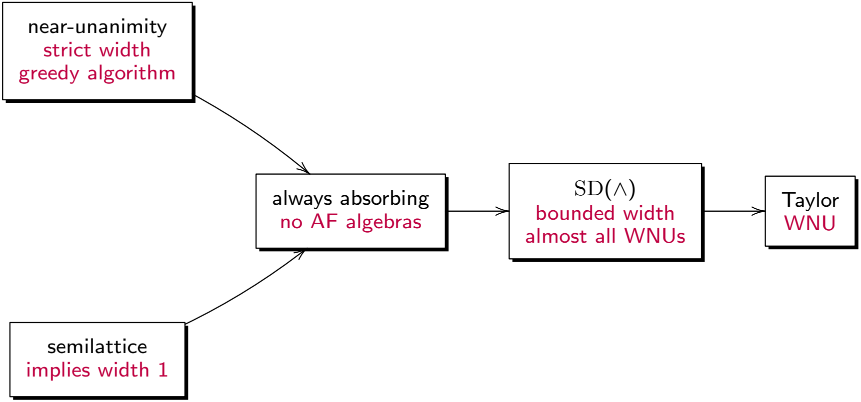
Absorption & always absorbing algebras

- an absorbing subuniverse ($C \trianglelefteq \mathbf{A}$) = there exists an idempotent $t \in \mathcal{F}$ such that

$$\begin{aligned} t(A, C, \dots, C, C) &\subseteq C, \\ t(C, A, \dots, C, C) &\subseteq C, \\ &\vdots \\ t(C, C, \dots, C, A) &\subseteq C. \end{aligned}$$

- an absorption-free (AF) algebra = no proper absorbing subuniverse
- an always absorbing (AA) algebra = every $C \leq \mathbf{A}$ has a proper absorbing subuniverse (equivalently, no AF algebra in $\mathrm{HSP}_{\mathrm{fin}}(\mathbf{A})$)
- example: NU, semilattice
- AA algebras are $SD(\wedge)$

Not every slide needs a title



CSP over digraphs aka $\mathbb{H}$-coloring problem

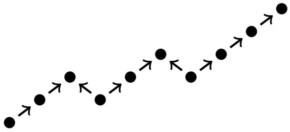
- Feder, Vardi '93: for every $\mathbb{A}$ there exists a digraph $\mathbb{H}$ such that $\mathtt{CSP}(\mathbb{A}) \overset{\mathtt{P}}{\sim} \mathtt{CSP}(\mathbb{H})$
- JB, Delić, Jackson, Niven '11: a simple construction, al(most al)l interesting Maltsev conditions are preserved, conjectures characterizing CSPs in P, NL, L reduce to digraphs news! actually, $\mathtt{CSP}(\mathbb{A}) \overset{\mathtt{L}}{\sim} \mathtt{CSP}(\mathbb{H})$ (talk to Marcel)
- why digraphs? fieldtest & inspiration for the algebraic approach, possibly interesting combinatorial facts
- Hell, Nešetřil '90: CSP dichotomy for undirected graphs
- Barto, Kozik, Niven '06: dichotomy for smooth digraphs in fact, core smooth Taylor digraph = disjoint union of directed cycles, thus has a majority $\Rightarrow$ is AA

Oriented trees

- oriented paths have both majority and semilattice $\Rightarrow$ are AA
- oriented triads (join 3 paths in one vertex) are already hard

Special oriented trees

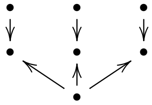
- oriented trees have levels; maximum level = height
- a minimal path = initial vertex has level 0, terminal vertex level k, and for all other vertices $0 < \mathrm{level}(v) < k$



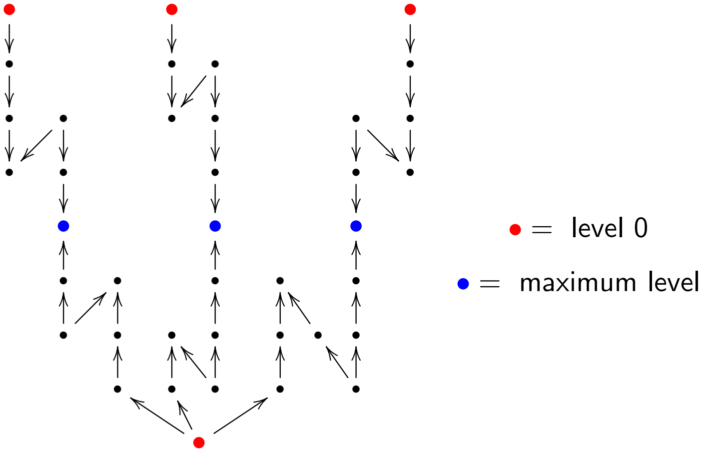
Definition

Let $\mathbb{T}$ be an oriented tree of height 1. A $\mathbb{T}$-special tree is an oriented tree obtained from $\mathbb{T}$ by replacing all edges by minimal paths of the same height (preserving orientation).

- a special triad = $\mathbb{T}$-special tree where

$\mathbb{T} =$ 

Example of a special triad



Barto, Kozik, Maróti, Niven: Is this the smallest NP-complete oriented tree? (38 vertices)

The history of special trees

- Gutjahr, Welzl, Woeginger '92: an NP-complete oriented tree (81 vertices)
- Hell, Nešetřil, Zhu '95: invented the special triads, constructed an NP-complete one (45 vertices) and more
- Barto, Kozik, Maróti, Niven '08: CSP dichotomy for special triads, Taylor implies either majority or width 1
- Barto, JB '10: CSP dichotomy for *special polyads*, Taylor implies $\mathrm{SD}(\wedge)$, a rather complicated proof

Theorem (JB '13)

The CSP (algebraic) dichotomy holds for all special trees. Taylor special trees are $\mathrm{SD}(\wedge)$. (Maybe even AA, work in progress. . .)

- an easy(-ish) proof, "localization", uses very recent algebraic tools

(I have no time for) sketch of the proof

- $\mathbb{H}$ – a $\mathbb{T}$-special tree, Taylor
  $\mathbb{T} = \langle A \cup B; E \rangle$, $E \subseteq A \times B$ – an oriented tree of height 1
- A, B and E are pp-definable from $\mathbb{H}$
- $\mathbb{H}$ is $SD(\wedge)$ iff both A and B are $SD(\wedge)$ (this is "special")
- A or B has a singleton absorbing subuniverse (Absorption theorem!)
- WLOG $\{o\} \trianglelefteq A$, partial ordering of $A \cup B$ by distance from o

- closer elements absorb more distant ones
- E-neighbourhoods of singletons are AA (this is the only technical bit; we construct nice binary polymorphisms)
- A and B are AA

Open problems & Thanks

Conjecture

Every Taylor oriented tree is already $SD(\wedge)$.
"Taylor trees cannot encode linear equations."

Problem

Is there a homotopy-like notion for oriented trees (cf. homotopy for reflexive digraphs of Larose and Tardif)?

Problem

Characterize (finite, idempotent) AA algebras.

Thank you for your attention!