Semigroups and Geometric Spaces

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Graphs

Let $S = \langle A \rangle$ be a semigroup.

Definition

We define the (right-)*Cayley graph* of S, $\mathrm{Cay}(S, A)$ to be:

- a set of vertices $V = S$
- a set of edges E
- a map $\iota : E \to V$
- a map $\tau : E \to V$
- a labelling map $l : E \to A$

where $(e)\iota = s$, $(e)\tau = s.a$ and $(e)l = a$.

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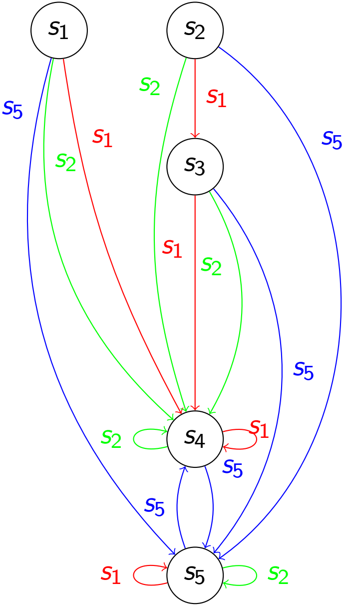
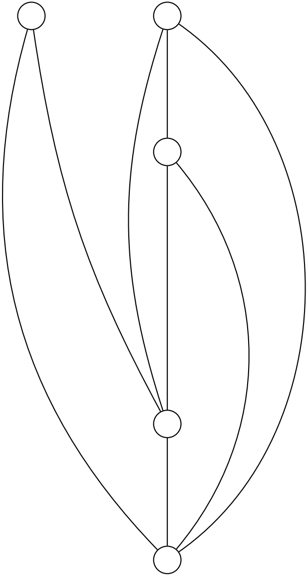
Definition

We define the skeleton $\dagger(S, A)$ of S to be:

- a set of vertices V
- a set of undirected edges $F = \{(\iota(e), \tau(e)), (\tau(e), \iota(e)) | e \in E, \iota(e) \neq \tau(e)\}$

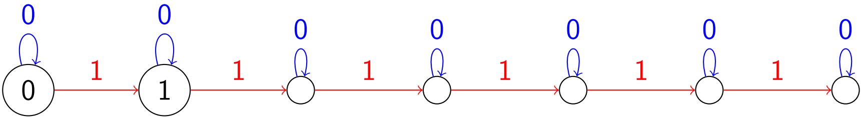
Graphs





Semigroup Properties

$\mathbb{N} \cup \{0\} = \langle 0, 1 \rangle$



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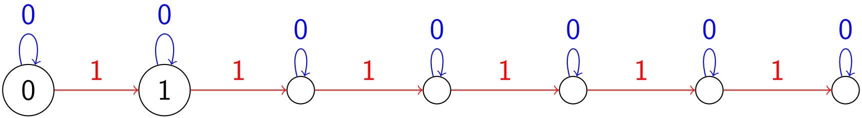

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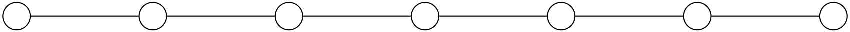

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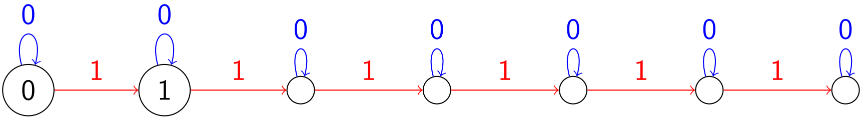

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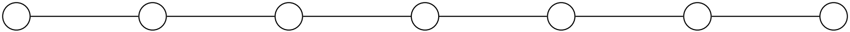
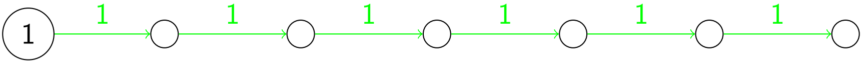


which is the skeleton for $\mathbb{N} = \langle 1 \rangle$ as well



Semigroup Properties

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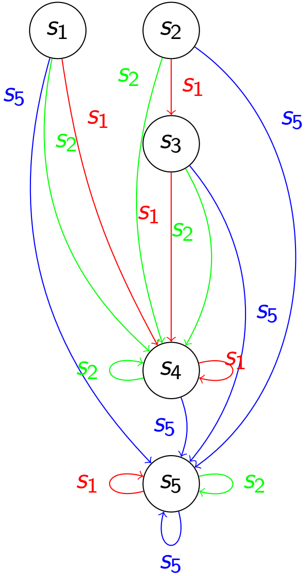

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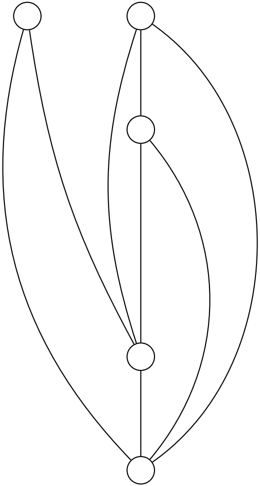

A Theorem from Groups

Theorem

Let $G = \langle A \rangle$ *and* $H = \langle B \rangle$ *be groups such that* $\dagger(G, A) \cong \dagger(H, B)$*. Then* G *is finitely presented if and only if* H *is.*

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Theorem

Let $G = \langle A \rangle$ and $H = \langle B \rangle$ be groups such that $\dagger(G, A) \cong \dagger(H, B)$. Then G is finitely presented if and only if H is.

Questions:

- How does this extend to semigroups?
- Does there exist a counterexample for semigroups?
- Are there any classes of semigroups where this property does hold?
- Given a particular $\dagger(S)$, can we determine which semigroup it represents?

An easy example

Definition

Let S be a semigroup and $l \in S$. Then l is called a *left zero* if for all $s \in S$, $l.s = l$. If all elements of a semigroup of size n are left zeros, we call this a left zero semigroup of size n, and denote it L_n. Right zero semigroups are defined analogously.

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Theorem

Let $S = G \times L_n = \langle A \rangle$ and $T = H \times L_m = \langle B \rangle$, and $\dagger(S, A) \cong \dagger(T, B)$. Then S is finitely presented if and only if T is.

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- $\dagger(S, A)$ is n disjoint copies of $\dagger(G, \pi_G(A))$
- $n = m$
- $\dagger(T, B)$ is n disjoint copies of $\dagger(H, \pi_H(B))$

A not-so-easy example

An obvious counterpart to the previous theorem would be semigroups of the type $G \times R_n$.

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Ideas/Problems:

- It would suffice to show that the groups are finitely presented.
- Considering the subgraph of $\mathrm{Cay}(S, A)$ with only vertices in $\{(g, r_1) | g \in G\}$, it is not necessarily true that this gives a copy of $\dagger(G, \pi_G(A))$.
- It is difficult to determine the set of vertices corresponding to $\{(g, r_1) | g \in G\}$ in $\dagger(S, A)$

Geometric Group Theory to the rescue!

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Theorem (Švarc-Milnor Lemma)

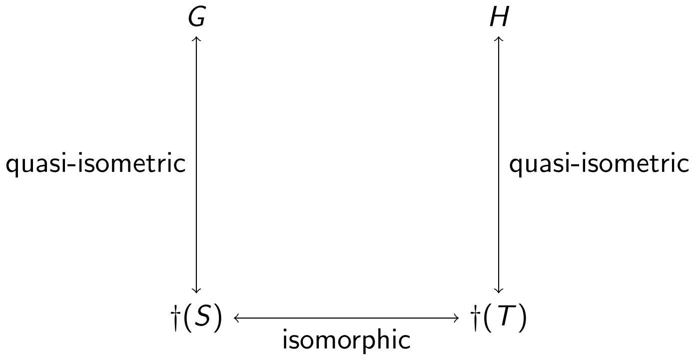
Let G be a group and X a proper geodesic metric space. Let G act properly and co-compactly by isometries on X. Then G is finitely generated and quasi-isometric to X. Moreover, for any $x \in X$, the mapping $G \to X$ given by $g \mapsto {}^{g}x$ is a quasi-isometry.

Geometric Group Theory to the rescue!

Let $S = G \times R_n$ and $T = H \times R_m$, and $\dagger(S) \cong \dagger(T)$

- Transform $\dagger(S)$ and and $\dagger(T)$ into sensible metric spaces.
- Act with G on $\dagger(S)$ by "left multiplication"
- Švarc-Milnor tells us G is quasi-isomteric to $\dagger(S)$

Geometric Group Theory to the rescue!



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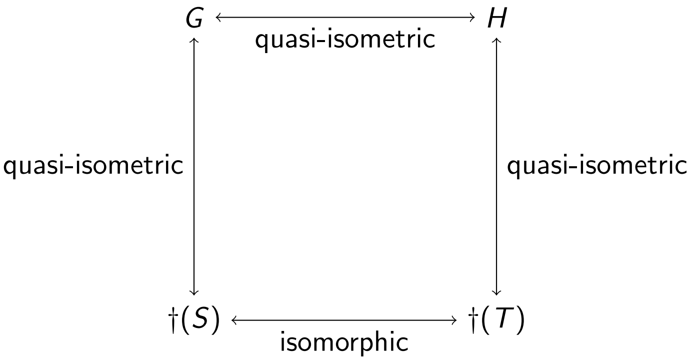

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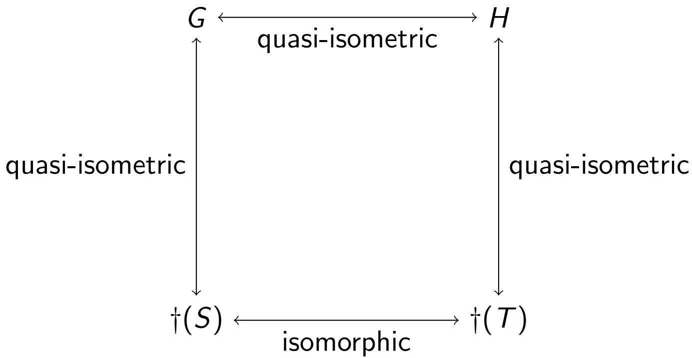

Finite presentability is preserved under quasi-isometries, so H and therefore T is finitely presented.

Completely Simple Semigroups

A finitely generated completely simple semigroup S can be thought of as a direct product $L_n \times G \times R_m$ with multiplication given by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$.

- $\dagger(S)$ has n disjoint components
- An action can be defined on $\dagger(S)$:

$$^{x}(i, g, \lambda) = (i, xg, \lambda)$$

- Apply Švarc-Milnor

Clifford Semigroups

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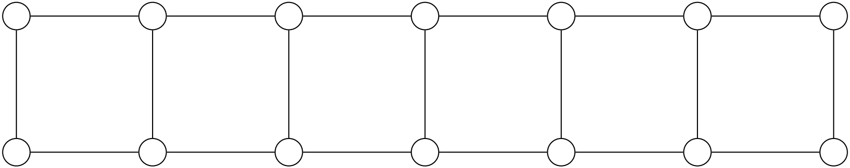

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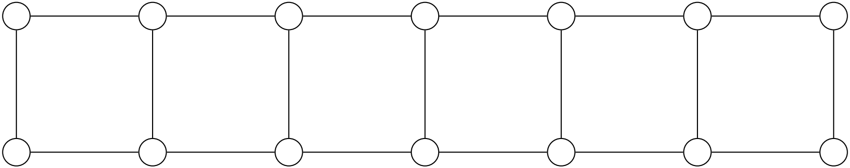

This can represent a Clifford semigroup with lattice size two: two copies of $\mathbb{Z}$ with the identity homomorphism. Or a Clifford semigroup with lattice size one: a single copy of $\mathbb{Z} \times C_2$.

Clifford Semigroups

A Small Example

Let $S = \langle A \rangle$ be a Clifford semigroup composed of two groups $G_1 \cong G_2$ with $\phi_{1,2}$ given by the isomorphism. Suppose that A is closed under inverses, and $A = A_1 \cup A_2$ where $A_1 \subseteq G_1$ and $A_2 \subseteq G_2$.
Then any vertex $v_1 \in G_1$ has degree $|A| + |A_1|$, and any vertex $v_2 \in G_2$ has degree $|A| + |A_2| - |A_1\phi_{1,2} \cap A_2|$.
Thus if $|A_1| \neq |A_1\phi_{1,2} \cap A_2|$, we can distinguish between vertices in G_1 and those in G_2.

Spectrum of $\dagger(S)$

Given a graph $\dagger(S)$, we might ask which S it could represent.

Definition

The *spectrum* of a semigroup $S = \langle A \rangle$ is the set of all semigroups T where $\dagger(S, A) \cong \dagger(T, B)$ for some generating set B of T. We denote this by $\sigma(S, A)$.

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Theorem (A,M. Quick,N. Ruškuc)

Let A^+ be the free semigroup on one generator. Then the spectrum of $A^+ = \langle 1 \rangle$ is $\sigma(A^+, 1) = \{A^+, A^\star\}$.

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Lemma

Let A^+ be the free semigroup on one generator and $S = \langle a, b \rangle$ be a semigroup such that $\dagger(S, \{a, b\}) \cong \dagger(A^+, 1)$. Then $S \cong A^\star$ or $S \cong A^+$.

Spectrum of $\dagger(S)$

Theorem

The spectrum of the integers is $\sigma(\mathbb{Z}, \{-1, 1\}) = \{\mathbb{Z}, C_2 \star C_2\}$.

Theorem

Let $A^\star$ *be the free monoid on* n *generators for* $n > 1$ *then* $\sigma(A^\star, \{a_1, \ldots, a_n\}) = \{A^\star\}$.