# Pseudovarieties generated by Brauer type monoids

K. Auinger

NSAC 2013

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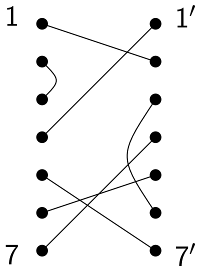

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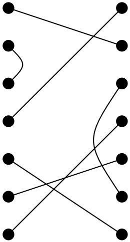

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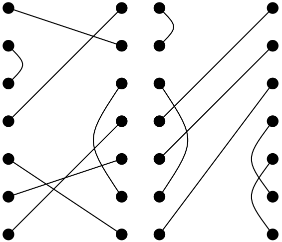

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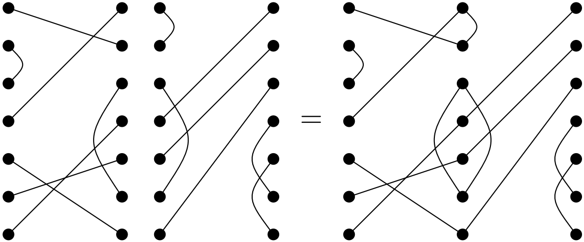

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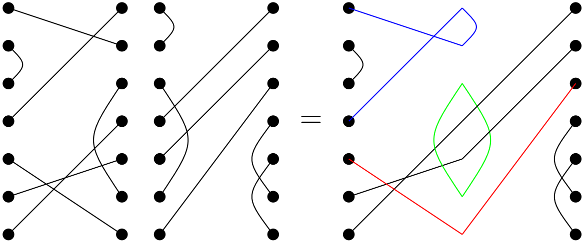

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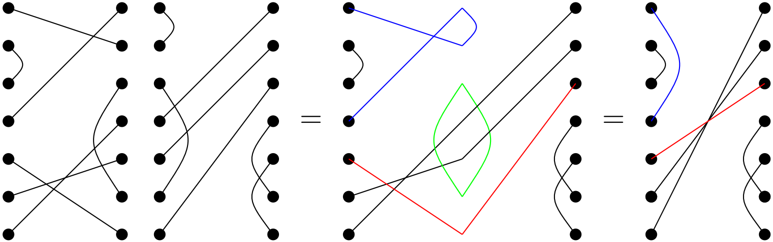

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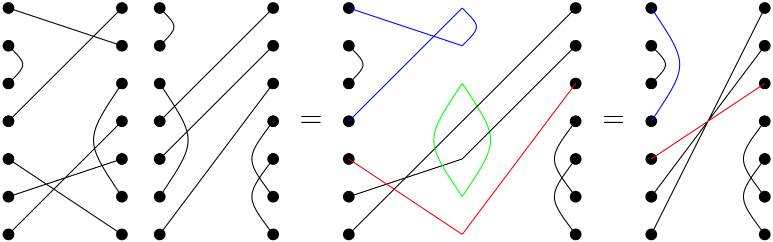

composition of diagrams defines a monoid structure on $\mathfrak{B}_n$, the *Brauer monoid*

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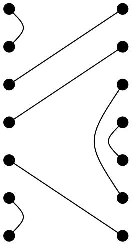

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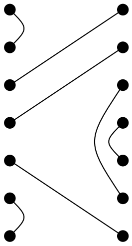

$\mathfrak{J}_n$ is closed under composition of diagrams, is called the *Jones monoid* or the *Temperley–Lieb monoid*

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- Almeida, Volkov (1998): $\{\mathscr{O}_n \mid n \in \mathbb{N}\}$ is far away from a generating series of $\mathbf{A}$: the interval $[\mathbf{O}, \mathbf{A}]$ where

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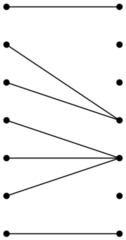

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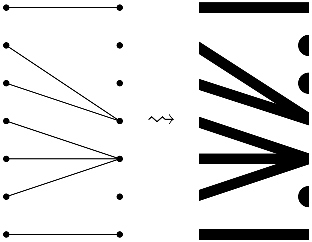

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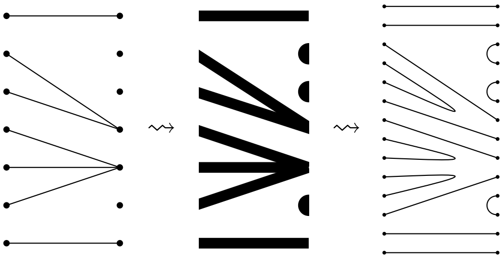

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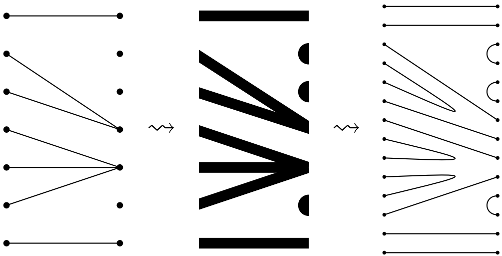

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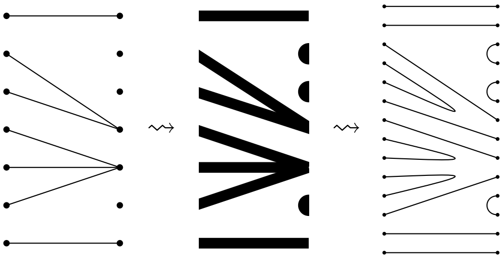

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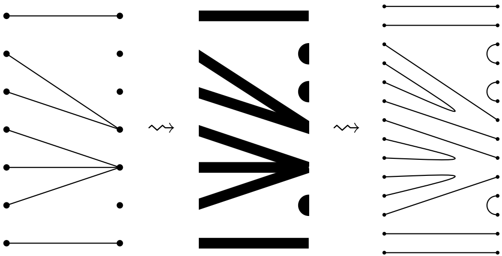

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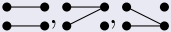
$\mathfrak{U}_2 := \{$  $\}$

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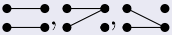

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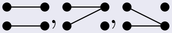
$\mathfrak{U}_2 := \{$  $\}$

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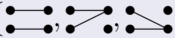
$$\mathfrak{U}_2 := \{$$  $$\}$$

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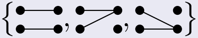
$$\mathfrak{U}_2 := $$ 

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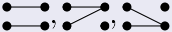
$\mathfrak{U}_2 := \{$  $\}$

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The claim then follows from the Krohn–Rhodes Theorem.

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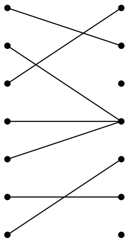

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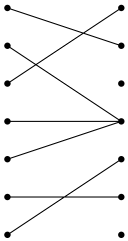

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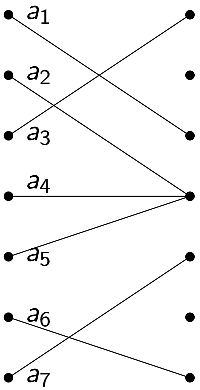


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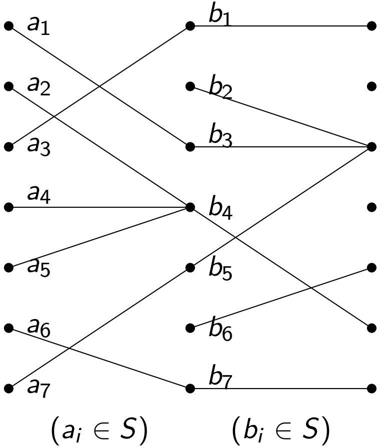

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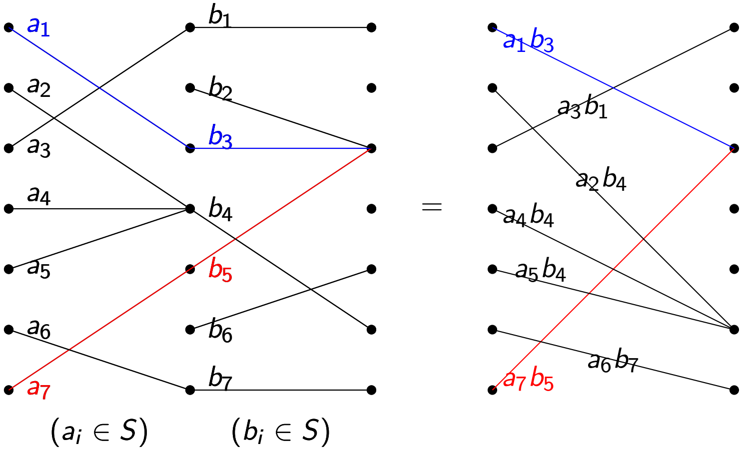

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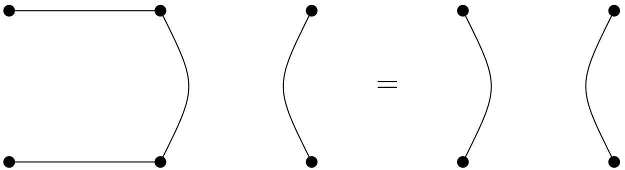

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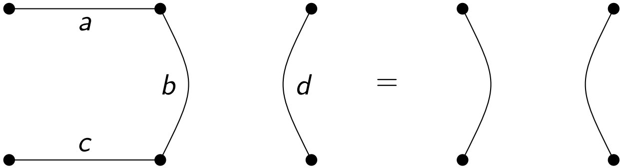

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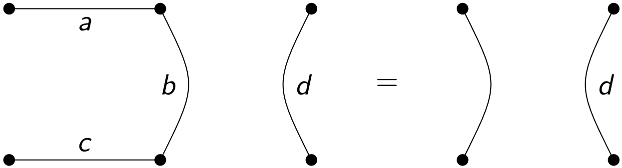

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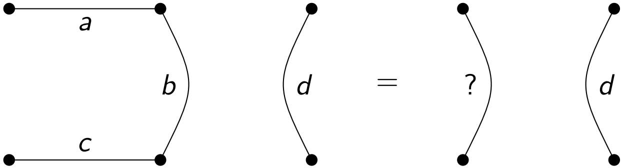

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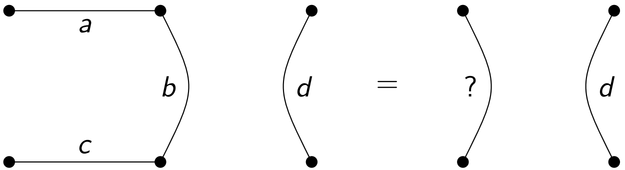


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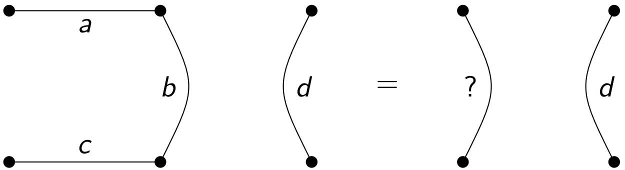


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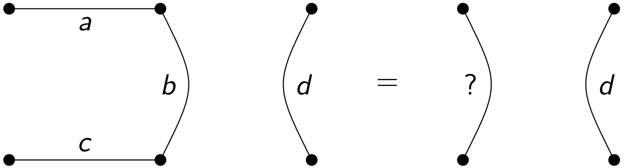
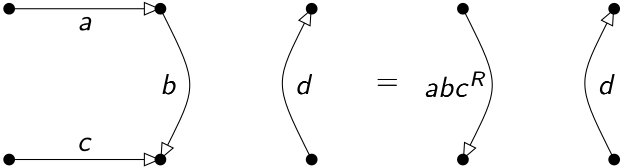



Solution: orient the arcs and label them by elements of a monoid with involution $x \mapsto x^R$!



$\rightsquigarrow$ *labelled product* $S \circledL \mathfrak{J}$ of an involutory monoid $S$ and $\mathfrak{J} \leq \mathfrak{B}_m$

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Essential properties:

$$\mathfrak{J}_n \textcircled{L} \mathfrak{J}_m \hookrightarrow \mathfrak{J}_{(n+2)m}$$

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Therefore,

$$S \prec \mathfrak{J}_n \Longrightarrow S \wr ([2], \mathfrak{U}_2) \prec \mathfrak{J}_n \circledL \mathfrak{J}_5 \hookrightarrow \mathfrak{J}_{5(n+2)}.$$

For the the claim

$$\mathrm{pvar}\{\mathfrak{B}_n \mid n \in \mathbb{N}\} = \mathbf{M}$$

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Thanks!